# Statistical Physics of Fields

## 1. Collective behavior, from particles to fields

Microcanonical ensemble assumes equal prob in each microstate

$$P_i \propto \Omega(E_{tot} - E_i)$$ specifying microstate $i$ has energy $E_i$

Entropy is additive but microstates are multiplicative

$$S = k \ln \Omega$$

Taylor expand entropy around a certain energy

$$S(E_{tot} - E_i) \approx S(E_{tot}) + \left(\frac{\partial S}{\partial E}\right) E_i$$

$$\downarrow$$

$$1/T$$

$$k \ln \Omega(E_{tot} - E_i) \propto e^{-\beta E_i}$$ Drop constant $S(E_{tot})$ term

Can also motivate this structure by

1) Independent Systems Factorize

$$P_{ij}^{AB} = P_i^A P_j^B$$

2) Energy is additive

$$E_{ij} = E_i + E_j$$

} Probability must be exponential

Operating under microcanonical framework

Exponential probability arises from additive constraints under independence

Logarithmic entropy comes from the microcanonical framework

Traditional Stat mech is the marriage of these two ideas

Goals of Course:

1) How can we characterize and understand the exciting behavior of macroscopic systems with strong interactions?

2) What are the characteristics of low energy excitations of the system?

[Fields arise from coarse graining microscopic degrees of freedom into smooth collective variables that capture the physics]

Translational symmetry makes the Hamiltonian a convolution and Fourier modes diagonalize convolutions

Energy is the expectation of the Hamiltonian operator

Locality: Assumes higher-order derivatives are less significant. Allows us to write potential as a function of local energy density

$$V[u] = \int dx\, \Phi(u(x), \partial u/\partial x, \ldots)$$

**Translational Symmetry**: Uniform translation doesn't change energy

$$\Phi[u(x)+c] = \Phi[u(x)]$$

Implies $\phi$ doesn't depend on $u(x)$ only its derivatives

**Stability**: Fluctuations are around equilibrium solution so there can be no linear terms in $u$ or its derivatives

$$V[u] = \int dx \left[ \frac{K}{2} \left(\frac{\partial u}{\partial x}\right)^2 + \frac{L}{2}\left(\frac{\partial^2 u}{\partial x}\right)^2 + \cdots + M\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right) + \cdots \right]$$

Can transform to Fourier Space

$$u(x) = \int \frac{dk}{2\pi} e^{ikx} u(k)$$

Apply the identity

$$\frac{d^n u}{dx^n} \longleftrightarrow (ik)^n u(k)$$

$$\int dx\, e^{i(k-k')x} = 2\pi \delta(k-k')$$

Straight forward math a lot of bookkeeping

Squared terms are over all fourier modes

The canonical partition function for a finite collection of particles is always analytic however singularities emerge in the thermodynamic limit

Phase transitions correspond to singularities in the free energy

Order parameters are thermodynamic functions that differ across phase

Divergences of response functions at a critical point imply correlated fluctuations over large distances

## 2. Statistical Fields

Consider the example of a Ferromagnet with a curie temperature $T_c$

The origin of the magnetism is microscopical and relies on the microscopic hamiltonian $\mathcal{H}_{mic}$

However, the scale of magnetic behavior around the curie point far exceeds this microscopic parameterization so we introduce intermediate mesoscopic scale in the form of a magnetization field $\vec{m}(x)$ (coarse graining)

$\vec{m}(x)$ is assumed to be constant on scale of lattice spacings $a$

Fourier transform involves wave vectors less than some cutoff $\Lambda \sim 1/a$

In this framework we compute the partition function over the coarse grained field

$$Z(T) = \sum e^{-\beta \mathcal{H}_{\text{microscopic}}} = \int \mathcal{D}\vec{m}(x)\, W[\vec{m}(x)]$$

↑ integrating over configurations

↑ probability weighting

Landau presented a formulation for $W[\vec{m}(x)]$ for the superfluidity of Helium which can be generalized for other phase transitions

Consider a generic $n$ component field existing in $d$-dimensional space

- $n=1$ corresponds to liquid-gas transitions
- $n=2$ applies to superfluidity, super conductivity and planar magnets
- $n=3$ corresponds to classical magnets
- $n=4$ is used for relativistic field theory

Based on the physics of the system we can encode structure onto the mean field boltzmann weights

## Locality and Uniformity

$$\beta \mathcal{H} = \int d^d x\ \phi[\vec{m}(x), x]$$

↑ energy density

↑ drop $x$ dependence if uniform in space

To encode non-local interactions we want to include higher order derivatives of $\vec{m}$

$$\beta \mathcal{H} = \int d^d x\ \phi[\vec{m}(x), \nabla\vec{m}, \nabla^2\vec{m}, \cdots]$$

This formulation is applicable for short range interactions

## Analyticity

Microscopic formulations are not clearly analytic (smooth + infinitely differentiable); however when we coarse grain and consider averages a CLT like argument implies that the probability distribution approaches a gaussian form.

This process washes out non-analyticities at the microscopic scale.

Non-analyticities at the mesoscopic scale only arise in the infinite limit

Finite sums of analytic functions are analytic

## Symmetry

Microscopic symmetry must be preserved in the mesoscopic perspective

Linear terms do not preserve rotational invariance

Gradients must be kept with regard to the spatial symmetries of the system

## Stability

Stability of the partition function requires free energy to be bounded from below

In practice this enforces the highest order terms to be positive so at large values $F \to \infty$

### Landau - Ginzberg Hamiltonian

$$\beta\mathcal{H} = \beta F_0 + \int d^d x \left[ \frac{t}{2} m^2(x) + u m^4(x) + \frac{K}{2} (\nabla m)^2 + \dots - \vec{h} \cdot \vec{m}(x) \right]$$

$\beta F_0$: Terms related to the non-magnetic degrees of freedom

$\vec{h}$: External magnetic field, $\vec{h} = \beta \vec{B}$

$\beta F_0$ is mostly ignored

The phenomenological parameters $t, u, K \dots$ have undetermined dependencies on microscopic interactions and external parameters

Computing the partition function of the LG Hamiltonian is difficult so we rely on approximations

Saddle-point integration replaces the integral with the maximum value of the integrand

this corresponds to the most probable configuration of the field

The $K(\nabla m)^2$ term penalizes spatial variation. From stability arguments we know $K > 0$.

Therefore we can argue that the most probable configuration has $(\nabla m)^2 = 0$

Attempting to minimize the exponent of the integrand

Equivalently, $\vec{m}(x)$ is uniform

$$Z \approx Z_{sp} = e^{-\beta F_0} \int d\vec{m} \exp\left[ -V \left( \frac{t}{2} + u m^4 + \dots - \vec{h} \cdot \vec{m} \right) \right]$$

$$\beta F_{sp} = -\ln Z_{sp} = -\beta F_0 + V \min\left[ \psi(\vec{m}) \right]$$

$$\psi(m) = \frac{t}{2} \vec{m}^2 + u(\vec{m}^2)^2 + \dots - \vec{h} \cdot \vec{m}$$

The most probable state is achieved when magnetization is aligned w/ external field

$$\vec{m} = m \hat{h}$$

The magnetization thus satisfies

$$\psi'(\bar{m}) = t\bar{m} + 4\bar{m}^3 + \dots - h = 0$$

The minimization of $\psi$ is not necessarily analytic

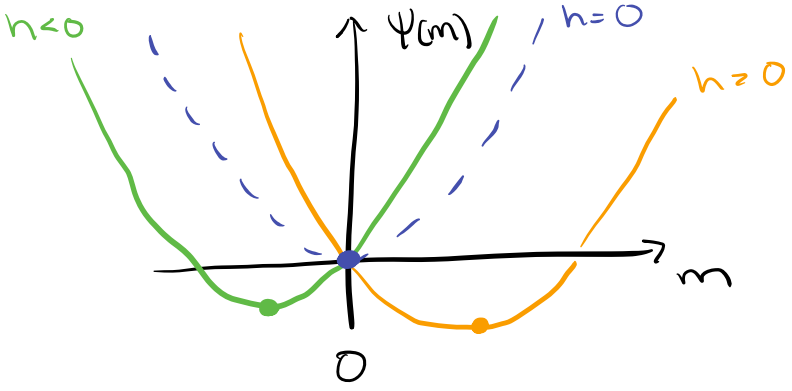
The behavior of $\psi$

Case 1: $t > 0$

Stability is enforced through quadratic term; can ignore quartic term

$$\min_{\bar{m}} [\psi(\bar{m})] = h/t$$

Corresponds to paramagnetic behavior

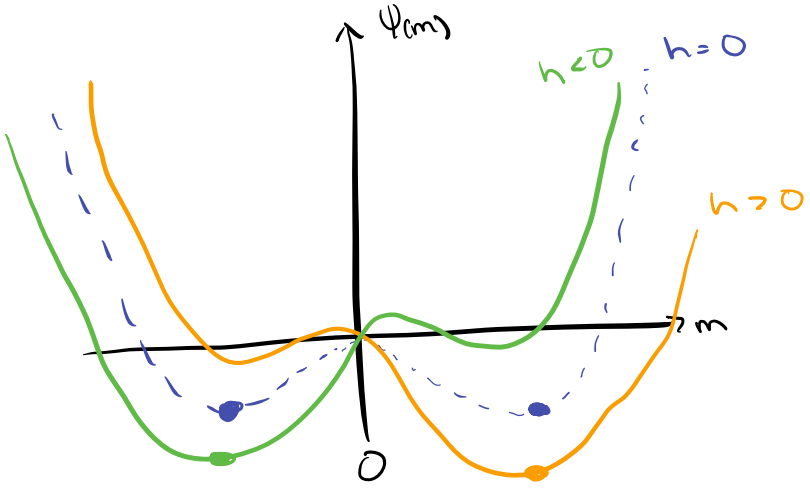



most probable magnetization occurs at the solid dots

Case 2: $t < 0$

Quartic term with positive $u$ is required to maintain stability

This introduces two local minima and a global minima

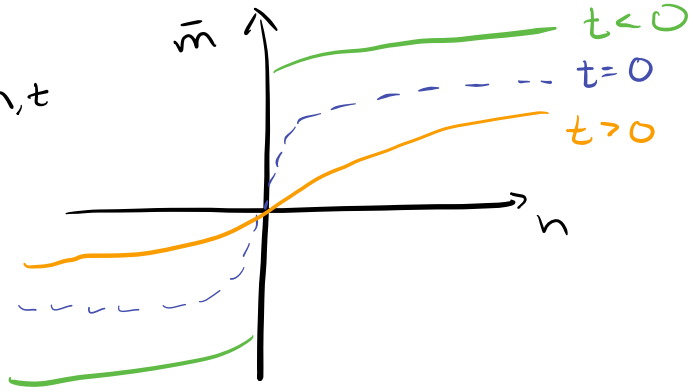


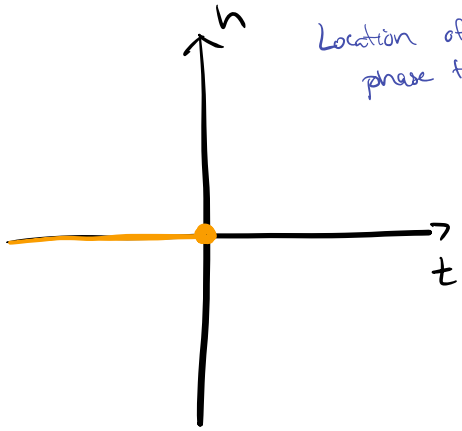

As $h \to 0$ the global minima is a non-zero value indicating spontaneous magnetization

Ferromagnetic

Direction of $m$ at $h = 0$ is history dependent

Optimal $\bar{m}$ for various $h, t$




Recall that each of the Landau-Ginzberg coefficients are analytic

We can correspondingly expand them around the critical point $T=T_c$

$$t = a_0 + a(T-T_c) + O(T-T_c)^2$$

$$u = u + u_1(T-T_c) + O(T-T_c)^2$$

$$K = K + K_1(T-T_c) + O(T-T_c)^2$$

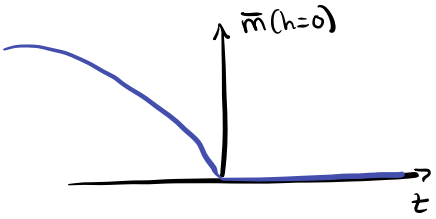
From experimental observations we can simplify the expansions to

$$t = a(T-T_c) + O(T-T_c)^2 \qquad (a,u,k) > 0$$

Predictions of Singular Behavior from Saddlepoint evaluation

Magnetization

When $h=0$, $\psi'(\bar{m}) = t\bar{m} + 4u\bar{m}^3 = \bar{m}(t + 4u\bar{m}^2)$




$$\bar{m}(h=0) = \begin{cases} 0 & t > 0 \\ \sqrt{-t/4u} = \sqrt{\frac{a}{4u}}\,(T-T_c)^{1/2} & t < 0 \end{cases}$$

Spontaneous magnetization vanishes with universal exponent $\boxed{\beta = 1/2}$

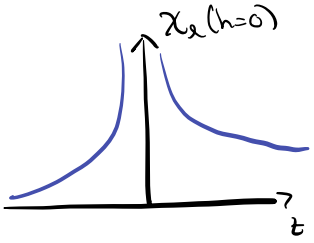
Amplitude is material dependent

When $t=0$, $\psi'(\bar{m}) = 4u\bar{m}^3 + h = 0$

$$\bar{m}(t=0) = \left(h/4u\right)^{1/3}$$

$\boxed{\delta = 3}$

## Susceptibility




Inverse susceptibility

$$\chi_2^{-1} = \left.\frac{\partial h}{\partial \bar{m}}\right|_{h=0} = t + 12u\bar{m}^2 = \begin{cases} t & t>0,\ h=0 \\ -2t & t<0,\ h=0 \end{cases}$$

Zero-field susceptibility diverges as we approach critical point

$$\chi_\pm \sim A_\pm |t|^{-\gamma_\pm} \qquad \boxed{\gamma = 1}$$

plug in $t = a(T-T_c) + \mathcal{O}(T-T_c)^2$

$$\frac{A_+}{A_-} = 2 \quad \text{universally}$$

## Heat Capacity

Free energy when $h=0$

$$BF = BF_0 + V \min_{\bar{m}} \{\psi(\bar{m})\}$$

$$\min \psi(\bar{m}) = \begin{cases} 0 & t>0 \\ -\frac{t^2}{8u} + \frac{t^2}{16u} = -\frac{t^2}{16u} & t<0 \end{cases}$$

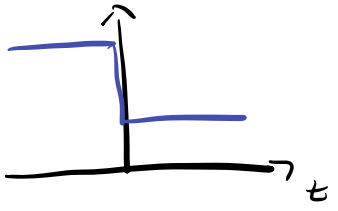
$$\frac{1}{2}t\bar{m}^2 + u\bar{m}^4$$

$$@\ \bar{m} = \sqrt{\frac{-t}{4u}}$$

Recall that $t = a(T-T_c) + \dots$ so $\frac{\partial}{\partial T} \sim a\frac{\partial}{\partial t}$

$$C(h=0) = -T\frac{\partial^2 F}{\partial T^2} \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(k_B T_c BF) = C_0 + V k_B a^2 T_c^2 \times \begin{cases} 0 & t>0 \\ 1/8u & t<0 \end{cases}$$

Saddle point predicts discontinuity rather than divergence



In systems with continuous symmetries there exist flat directions in energy which allow for cost free fluctuations known as goldstone modes

## 3. Fluctuations

We previously saw that we can study magnetization by considering the most probable state of the Landau-Ginzberg Hamiltonian

$$\mathcal{P}[m(\vec{x})] \propto \exp\left\{ -\int d^d x \left[ \frac{\kappa}{2} (\nabla m)^2 + \frac{t}{2} m^2 + u m^4 \right] \right\}$$

The most probable state corresponds to uniform $m(x) = \bar{m}\hat{e}_1$

We can think about fluctuations around this configuration in terms of directional components

$$\vec{m}(x) = \left[ \bar{m} + \phi_l(x) \right] \hat{e}_1 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}(x)\, \hat{e}_\alpha$$

$\phi_l$: longitudinal; $\phi_{t,\alpha}$: transverse

$$\phi_t = \sum_{\alpha=2}^{n} \phi_{t,\alpha}\, \hat{e}_\alpha$$

The terms in the LG Hamiltonian now take the form

$$(\nabla m)^2 = \nabla m \cdot \nabla m = \sum \left( \frac{\partial m}{\partial x_i} \right)^2 = (\nabla \phi_l)^2 + (\nabla \phi_t)^2$$

$$m^2 = \bar{m}^2 + 2\bar{m}\phi_l + \phi_l^2 + \phi_t^2$$

$$m^4 = \bar{m}^4 + 4\bar{m}^3\phi_l + 6\bar{m}^2\phi_l^2 + 2\bar{m}^2\phi_t^2 + \mathcal{O}(\phi_l^3, \phi_t^3)$$

Now, in our Hamiltonian

$$\beta\mathcal{H} = -\ln \mathcal{P} = V\left( \frac{t}{2}\bar{m}^2 + u\bar{m}^4 \right) + \int d^d x \left[ \frac{\kappa}{2} (\nabla\phi_l)^2 + \left( t\bar{m} + 4u\bar{m}^3 \right)\phi_l \right.$$

integrate out constant contributions

$$\left. + \left( \frac{t}{2} + 6u\bar{m}^2 \right)\phi_l^2 \right] + \int d^d x \left[ \frac{\kappa}{2} (\nabla\phi_t)^2 + \left( \frac{t}{2} + 2\bar{m}^2 u \right)\phi_t^2 \right] + \mathcal{O}(\phi_l^3, \phi_t^3)$$

We drop linear terms based on the observation $\bar{m}$ is an extrema

$\left(t\bar{m} + 4\bar{m}^3 u\right)\phi_q$ is the extrema condition!

We define restoring potentials

$$\frac{\kappa}{\xi_q^2} \equiv t + 12u\bar{m}^2 = \frac{\partial^2\psi}{\partial\phi_q^2}\bigg|_{\bar{m}} = \begin{cases} t & t>0 \\ -2t & t<0 \end{cases}$$

From case work previously

$$\frac{\kappa}{\xi_t^2} \equiv t + 4u\bar{m}^2 = \frac{\partial^2\psi}{\partial\phi_t^2}\bigg|_{\bar{m}} = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

Only see distinction when $t<0$

Rotating transverse modes provides no energy cost (Goldstone modes)

Now consider a Fourier change of variables

$$\phi(x) = \sum_q \phi_q e^{iq\cdot x} \cdot 1/\sqrt{V}$$

$$\nabla\phi(x) = \sum_q (iq)\phi_q e^{iqx} \cdot 1/\sqrt{V}$$

Gradient term:

$$(\nabla\phi(x))^2 = \sum_q \sum_{q'} (iq)(iq')\phi_q\phi_{q'} e^{iqx} e^{iq'x} \cdot 1/V$$

When we integrate over space we apply the orthogonality property of fourier modes to kill cross terms

$$\int (\nabla \phi(x))^2 d^d x = \frac{1}{V} \sum_{q,q'} (-q \cdot q') \phi_q \phi_{q'} \int d^d x \; e^{i(q+q')x}$$

$\downarrow$ $q' = -q$ survives

$$= \sum_q q^2 \phi_q \phi_{q'}$$

For a real field we enforce $\phi_{-q} = \phi_q^*$

think of needing to cancel imaginary parts of $\pm q$ mode

$$= \sum_q q^2 |\phi_q|^2$$

Normal term :

$$\int d^d x \; \phi^2 = \int d^d x \cdot \frac{1}{V} \cdot \sum_{q,q'} \phi_q \phi_{q'} e^{i(q+q')x}$$

$$= \frac{1}{V} \sum_{q,q'} \phi_q \phi_{q'} \underbrace{\int d^d x \, e^{i(q+q')x}}_{V \delta_{q,-q}}$$

$$= \sum_q |\phi_q|^2$$

Putting these together the Fourier basis form of the LG Hamiltonian w/ fluctuations

$$\beta H = V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \frac{K}{2}\left(\sum_q q^2 |\phi_{\alpha,q}|^2 + \frac{1}{\xi_\alpha^2}\sum_q |\phi_{\alpha,q}|^2\right)$$

$$+ \frac{K}{2}\left(\sum_q q^2 |\phi_{b,q}|^2 + \frac{1}{\xi_t^2}\sum_q |\phi_{b,q}|^2\right)$$

Consequently,

$$P \propto e^{-\beta H} = \prod_q \exp\left\{-\frac{K}{2}\left(q^2 + \xi_\alpha^{-2}\right)|\phi_{\alpha,q}|^2\right\} \exp\left\{-\frac{K}{2}\left(q^2 + \xi_t^{-2}\right)|\phi_{b,q}|^2\right\}$$

The probability distribution factorizes over nodes implying that each node is independent.

We also observe that the random variables are mean 0 gaussian distributed

with variance $1/K(q^2+\xi_\alpha^{-2})$

The covariance across nodes also factorizes $\langle \phi_{\alpha,q}\phi_{\beta,q'}\rangle = \langle\phi_{\alpha,q}\rangle\langle\phi_{\beta,q'}\rangle$

We know this to be 0 mean unless $\alpha=\beta$, $q=q'$

Therefore our correlation is given by

$$\langle \phi_{\alpha,q}\phi_{\beta,q'}\rangle = \frac{\delta_{\alpha,\beta}\delta_{q,q'}}{K(q^2+\xi_\alpha^{-2})}$$

This correlation structure is known as the Lorentzian form

This gives us fluctuation behavior as a function of mode and correlation length

In real space,

$$\langle \phi_\alpha(x)\,\phi_\beta(x')\rangle = \left\langle \frac{1}{V}\sum_{q,q'} \phi_{\alpha,q}(x)\, e^{iqx}\, \phi_{\beta,q'}(x')\, e^{iq'x'} \right\rangle$$

$$= \frac{1}{V}\sum_{q,q'} e^{iqx+iq'x'} \langle \phi_{\alpha,q}(x)\,\phi_{\beta,q'}(x')\rangle$$

$$= \frac{\delta_{\alpha,\beta}}{V}\sum_{q} e^{iqx-iqx'} \cdot \frac{1}{K(q^2+\xi_\alpha^{-2})}$$

We define

$$I_d(x-x',\xi_\alpha) = -\frac{1}{V}\sum_q \frac{e^{iq(x-x')}}{K(q^2+\xi_\alpha^{-2})}$$

$\to$ Continuum Limit

$$= -\int \frac{d^dq}{(2\pi)^d}\frac{e^{iq(x-x')}}{q^2+\xi_\alpha^{-2}}$$

↑ density of Fourier modes with periodic boundaries

We make the observation that $I_d$ must obey the following differential eq

$$\nabla^2 I_d(x,\xi) = -\int \frac{d^dq}{(2\pi)^d}\frac{\nabla^2 e^{iqx}}{(q^2+\xi^{-2})} = \int \frac{d^dq}{(2\pi)^d}\frac{q^2 e^{iqx}}{(q^2+\xi^{-2})} = \int \frac{d^dq}{(2\pi)^d} e^{iqx}\left[1-\frac{\xi^{-2}}{(q^2+\xi^{-2})}\right]$$

Recall $\int \frac{d^dq}{(2\pi)^d} e^{iq\cdot x} = \delta^d(x)$

$$= \delta^d(x) + \frac{I_d(x)}{\xi^2}$$

# Green's Functions

For any arbitrary system governed by a linear operator

$$L f(x) = \delta(x)$$

field source

The green's function of an operator is the systems response to a point source

$$L G(x,x') = \delta(x-x')$$

Solutions for any linear equation can be constructed from a superposition of point sources

$$s(x) = \int s(x') \delta(x-x') dx' = \int s(x') L G(x,x') dx'$$

$$f(x) = L^{-1} s(x) = \int G(x,x') s(x') dx'$$

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$$\nabla^2 I_d(x) = \delta^d(x) + \frac{I_d}{\xi^2}$$

From the differential equation we observe that the solution must be rotationally invariant

$\nabla^2$ is rotationally invariant

We can then parameterize the expression in terms of $r$

$$\frac{\partial f}{\partial x_i} = f'(r) \frac{\partial r}{\partial x_i} = f'(r) \frac{x_i}{r}$$

$$r = \sqrt{x_1^2 + \dots x_n^2} \rightarrow \frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{1}{r} \cdot 2x_i = \frac{x_i}{r}$$

$$\frac{\partial^2 f}{\partial x_i^2} = f''(r) \cdot \frac{\partial r}{\partial x_i} \cdot \frac{\partial r}{\partial x_i} + f'(r) \cdot \frac{\partial^2 r}{\partial x_i^2}$$

$$= f''(r) \cdot \frac{x_i^2}{r^2} + f'(r) \cdot \left(1/r - x_i^2/r^3\right)$$

$$\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = 1/r + x_i \cdot -1/r^2 \cdot \frac{\partial r}{\partial x_i}$$

$$= \frac{1}{r} - \frac{x_i^2}{r^3}$$

$$\nabla^2 f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} = f''(r) + f'(r)\left(\sum_{i=1}^{n} 1/r - \sum_{i=1}^{d} \frac{x_i^2}{r^3}\right)$$

$$= f''(r) + f'(r)\left(d/r - 1/r\right)$$

$$\frac{d^2 I_d(r)}{dr^2} + \frac{d\, I_d(r)}{dr}\left(\frac{d-1}{r}\right) - \frac{I_d}{\xi^2} = \delta(r)$$

Reduced dimensionality of problem

We consider a solution that decays exponentially at large distances

$$I_d(x) \propto \frac{\exp(-x/\xi)}{x^p}$$

$$\frac{dI_d}{dr} = -\left(\frac{p}{r} + 1/\xi\right) I_d$$

$$\frac{d^2 I_d}{dr^2} = \left(\frac{p(p+1)}{r^2} + \frac{2p}{r\xi} + 1/\xi^2\right) I_d$$

We can plug these into the differential equation condition

$$\frac{p(p+1)}{x^2} + \frac{2p}{x\xi} + \frac{1}{\xi^2} - \frac{p(d-1)}{x^2} - \frac{(d-1)}{x\xi} = 1/\xi^2$$

Implies $p(p+1) = p(d-1)$ and $p = d-2$

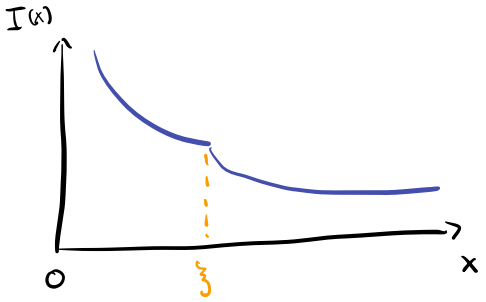
This is the exponent of Coulomb interactions!

The normalized result is thus

$$I_d(x) \simeq C_d(x) = \frac{x^{2-d}}{(2-d)\, S_d} \qquad x \ll \xi$$

$$1/x\xi \qquad x \gg \xi$$

$$\xi^{(3-d)/2} / (2-d) S_d\, x^{(d-1)/2} \exp(-x/\xi) \qquad x \simeq \xi \leftarrow \text{matching scales}$$




This tells us that the behavior of transverse and longitudinal correlations behave differently.

Transverse correlation length is infinite for $t < 0$ but matches $\xi$ for $t > 0$

Mermin-Wagner Theorem: No spontaneous breaking of continuous symmetry in systems with short range interactions in dimensions $d \leq 2$

(Maybe review this section)

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## Gaussian Integrals

In one dimension the gaussian integral is

$$I_1 = \int_{-\infty}^{\infty} d\phi\, e^{-K/2\,\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{h^2/2K}$$

$-\frac{K}{2}\left(\phi - \frac{h}{K}\right)^2 + h^2/K^2$ ← complete the square

As we take derivatives wrt to $h$

$$\frac{d}{dh} : \int_{-\infty}^{\infty} d\phi\, \phi\, e^{-K/2\,\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{h^2/2K} \cdot h/K$$

$$\frac{d^2}{dh^2} : \int_{-\infty}^{\infty} d\phi\, \phi^2 e^{-K/2\,\phi^2 + h\phi} = \sqrt{\frac{2\pi}{K}}\, e^{h^2/2K} \left[1/K + h^2/K^2\right]$$

These are the first and second moments of the pdf (after accounting for normalization)

Cumulants are given by

$$K = \ln(M) = \ln\left(\int_{-\infty}^{\infty} d\phi\, e^{-K/2\,\phi^2 + h\phi}\right) = \ln\left(\sqrt{\frac{2\pi}{K}}\right) + \frac{h^2}{2K}$$

First Cumulant: $\frac{\partial K}{\partial h} = {}^{h}/_{K}$

Second Cumulant: $\frac{\partial^2 K}{\partial h^2} = {}^{1}/_{K}$

Higher order cumulants are 0!

Can show this generally by considering

$K(h) = \ln M(h)$

$K'(h) = \frac{M'(h)}{M(h)}$

$K''(h) = \frac{M''(h)}{M(h)} - \frac{M'(h)^2}{M(h)^2}$

$\vdots$

All higher order cumulants of gaussian rv are 0

$K(h) = \ln M(h)$

Define the cumulants as $\kappa_\ell = \left.\frac{d^\ell K}{dh^\ell}\right|_{h=0}$

$$K(h) = \sum_{\ell=0}^{\infty} \frac{h^\ell}{\ell!} \kappa_\ell \leftarrow \text{Taylor Expansion}$$

so generically,

$$M(ik) = \langle e^{ik\phi_i} \rangle = \exp\left\{ \sum_{\ell=1}^{\infty} \frac{(ik)^\ell}{\ell!} \kappa_\ell \right\} = \exp\left[ ik \cdot \frac{h}{K} + \frac{(ik)^2}{2} \cdot {}^{1}/_{K} \right] = \exp\left[ ihk - \frac{k^2}{2K} \right]$$

$\rightarrow$ definition of MGF

$\rightarrow$ $M(h) = \exp\{K(h)\}$

The cumulant expansion gives us a simple expression for the MGF of gaussian distribution

Under gaussian fluctuations only pairwise correlations exist

Now, let's consider a more general case of a gaussian integral involving N variables

$$\mathcal{I}_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left\{ -\sum_{ij} \frac{K_{ij}}{2} \phi_i \phi_j + \sum_i h_i \phi_i \right\}$$

We can simplify the integral by diagonalizing $K$ with eigenvectors $\hat{q}$ and eigenvalues $K_\ell$

$\phi_i = \sum_\ell \tilde{\phi}_\ell \hat{q}_i$ $\leftarrow$ translation between space and eigencoordinates

Jacobian of this transformation is unitary (since $K$ is symmetric)

$$\mathcal{I}_N = \prod_{q=1}^{N} \int_{-\infty}^{\infty} \underbrace{|\det K|}_{1} d\tilde{\phi}_\ell \exp\left[ -\frac{K_\ell}{2} \tilde{\phi}_\ell^2 + \tilde{h}_\ell \tilde{\phi}_\ell \right] = \prod_{q=1}^{N} \sqrt{\frac{2\pi}{K_\ell}} \exp\left\{ \frac{\tilde{h}_\ell K^{-1} \tilde{h}_\ell}{2} \right\}$$

All constants

Fourier transform also preserves measure

Recall that $\prod_\ell K_\ell = \det K$

$$= \sqrt{\frac{(2\pi)^N}{\det(K)}} \exp\left\{ \sum_{ij} \frac{K^{-1}_{ij}}{2} h_i h_j \right\}$$

some nice linear algebra

$$\sum_q \frac{\tilde{h}_q}{2k_q} = \frac{1}{2} \tilde{h}^T D^{-1} \tilde{h}$$

$$\tilde{h} = U h$$

$$= \frac{1}{2} h^T K^{-1} h$$

The associated cumulants

$$\begin{cases} \langle \phi_i \rangle_c = \sum_j K^{-1}_{ij} h_j \\ \langle \phi_i \phi_j \rangle_c = K^{-1}_{ij} \end{cases}$$

We can thus write

$$\langle \exp(A) \rangle = \exp\left[ \langle A \rangle_c + \frac{1}{2} \langle A^2 \rangle_c \right]$$

for $A = \sum_i a_i \phi_i$

Next we can consider gaussian functional integrals

$\phi_i \to \phi(x)$

$K_{ij} \to K(x,x')$ ← kernel

$$\int_{-\infty}^{\infty} \mathcal{D}\phi(x) \exp\left\{ - \int d^d x \, d^d x' \, \frac{K(x,x')}{2} \phi(x) \phi(x') + \int d^d x \, h(x) \phi(x) \right\}$$

$$\propto (\det K)^{-1/2} \exp\left[ \int d^d x \, d^d x' \, \frac{K^{-1}(x,x')}{2} h(x) h(x') \right]$$

where

$$\int d^d x' \, K(x,x') \, K^{-1}(x', x'') = \delta^d(x - x'')$$

The cumulant expression generalizes to

$$\begin{cases} \langle \phi(x) \rangle_c = \int d^d x' \, K^{-1}(x,x') h(x') \\ \langle \phi(x) \phi(x') \rangle_c = K^{-1}(x,x') \end{cases}$$

← desired correlation functions

Correlation Function = Inverse of Quadratic Operator = Green's Function

---

Now, returning to the saddle point approximation with gaussian fluctuations

$$Z \approx \exp\left[ -V\left( \frac{t}{2} \bar{m}^2 + u \bar{m}^4 \right) \right] \int \mathcal{D}\phi_\ell \exp\left\{ -\frac{K}{2} \int d^d x \left[ (\nabla \phi_\ell)^2 + \frac{\phi_\ell^2}{\xi_\ell^2} \right] \right\}$$

$$\cdot \int \mathcal{D}\phi_t \exp\left\{ -\frac{K}{2} \int d^d x \left[ (\nabla \phi_t)^2 + \frac{\phi_t^2}{\xi_t^2} \right] \right\}$$

Gaussian kernel diagonalizes in Fourier space with eigenvalues $K(q) = K(q^2 + \xi^{-2})$

$$\ln \det K = \sum_q \ln K(q) = \int \frac{d^d q}{(2\pi)^d} \ln\left[K(q^2 + \xi^{-2})\right]$$

The free energy is consequently

$$\beta f = -\frac{\ln Z}{V} = \frac{t\bar{m}^2}{2} + u\bar{m}^4 + \frac{1}{2}\int \frac{d^d q}{(2\pi)^d} \ln\left[K(q^2 + \xi_+^{-2})\right] + \frac{n-1}{2}\int \frac{d^d q}{(2\pi)^d} \ln\left[K(q^2 + \xi_-^{-2})\right]$$

If we consider the heat capacity we get correction terms proportional to

$$C_{\text{singular}} \propto -\frac{\partial^2(\beta f)}{\partial^2 t} = \begin{cases} \ldots + I_+ C_F & t > 0 \\ \ldots + I_- C_F & t < 0 \end{cases}$$

$$C_F = \frac{1}{K^2}\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \xi^{-2})^2}$$

This has effective dimension $L^{4-d}$

$$\left.\begin{aligned} q &= L^{-1} \\ d^d q &= L^{-d} \end{aligned}\right\} \quad (q^2 + \xi^{-2})^2 \to L^{-4} \to \text{Integrand} \sim L^{4-d}$$

Therefore we find $d = 4$ as the <u>upper critical dimension</u>

$d \geq 4$ : Short distance physics dominates since the integrand diverges for large $q$ (short range)

Dominated by upper cutoff $\Lambda \sim 1/a$

This implies that the saddle point approximation is appropriate

Fluctuation correction remains finite as $\xi \to \infty$ (main contribution being $q^2$ term)

$d < 4$ : This diverges as $\xi \to \infty$ ($\xi \to \infty$ at criticality)

so mean field breaks

Validity of fluctuation corrections is dependent on their correlation length dependence

<u>Ginzburg Criterion</u> is a quantitative test when mean-field (saddlepoint approximation + gaussian fluctuation) is self-consistent

Mean field finds $t < T_c$ $\quad \bar{m}^2 \sim \frac{|t|}{u}$ (saddlepoint)

$m(x) = \bar{m} + \phi(x)$ (fluctuations)

Require $\langle \phi^2 \rangle \ll \bar{m}^2$

$$|t| \ll t_G \simeq \frac{1}{\left(\xi_0^d \Delta C_{sp}\right)^{\frac{2}{4-d}}}$$

ginzburg temperature for which mean-field is appropriate

Skipped Derivation

Upper critical Dimension 4: Saddlepoint approximation is valid

Lower critical Dimension: Fluctuations are strong enough to destroy ordered phase

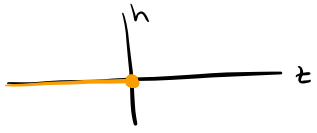
1 discrete symmetry
2 continuous symmetry

Intermediate dimensions are perturbed from saddlepoint but order still exists

## 4. Scaling Hypothesis

We have explored various scaling behaviors in the vicinity of the critical point. These quantities are necessarily related so the goal of this section is to understand the minimal set of exponents necessary to characterize observed behavior

Consider the behavior of a magnet in a field. Recall the phase diagram




The free-energy via saddlepoint approximation is given by

$$f(t,h) = \min\left[\frac{t}{2}m^2 + um^4 - hm\right]_m = \begin{cases} -1/16\ t^2/u & h=0,\ t<0 \\ -3/4^{4/3}\ \dfrac{h^{4/3}}{u^{1/3}} & h\neq 0,\ t=0 \end{cases}$$

Assume that the singular parts of free energy can be written as a single homogeneous function of $t$ and $h$

$$f(t,h) = |t|^2\, g_f\left(h/|t|^{\Delta}\right)$$ $\Delta$ is the gap exponent

Homogeneous function implies that $f(b^{p_1}x, b^{p_2}x, \dots) = b^{p_f} f(x, y z_1, \dots)$

Not entirely sure why we make this assumption but it is related to scale invariance around critical point

We can then match this behavior to our known $f(t,h)$

$$\lim_{x\to 0} g_f(0) \sim 1/u$$

$$\lim_{x\to\infty} g_f(\infty) \sim \frac{x^{4/3}}{u^{1/3}}$$

$$f_{sing} \sim \frac{|t|^2}{t^{4\Delta/3}}\frac{h^{4/3}}{u^{1/3}} \longrightarrow \frac{6-4\Delta}{3} = 0$$

$$\Delta = 3/2$$

We assume that this homogeneous structure is retained beyond the saddlepoint

$$f_{sing}(t,h) = |t|^{2-\alpha}\, g_f\left(h/t^{\Delta}\right)$$

taking this on faith (perhaps there is strong motivation)

$\alpha$ and $\Delta$ change based on the critical point in consideration

The structure is chosen such that

$$C_{sing} \sim -\frac{\partial^2 f}{\partial t^2} \sim |t|^{-\alpha} g_c(h/|t|^{\Delta})$$

Heat capacity scaling behavior as $h \to 0$ $\quad \sim |t|^{-\alpha}$

Exponents for $\pm t$ must align

Argument relying on analyticity of free energy except on coexistence line

Based on our free energy scaling hypothesis we can recover singular behavior for other quantities of interest

Magnetization

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta} g_m(h/|t|^{\Delta})$$

In the limit $h \to 0$

$$m(t,h=0) \sim |t|^{\beta} \qquad \beta = 2-\alpha-\Delta$$

In the limit $x \to \infty$, $g_m(x) \sim x^p$

$$m(t=0,h) \sim |t|^{2-\alpha-\Delta} \left(\frac{h}{|t|^{\Delta}}\right)^p$$

this implies that $\quad p\Delta = 2-\alpha-\Delta$

$$m(t=0,h) \sim h^{\frac{2-\alpha-\Delta}{\Delta}} \qquad \delta = \frac{\Delta}{2-\alpha-\Delta} = \frac{\Delta}{\beta}$$

Susceptibility

$$\chi(t,h) = \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta})$$

$$\chi(t,h=0) \sim |t|^{2-\alpha-2\Delta} \qquad \gamma = 2\Delta - 2 + \alpha$$

These calculations show that the same $\Delta$ is shared across thermodynamic singularities with all scaling behavior fully defined by 2 independent values

At the critical point we know correlation length diverges. We show this via the divergence of the thermodynamic quantities above

To actually quantify the scaling behavior of $\xi$ near criticality we introduce two assumptions

1) Correlation length is a homogeneous function

$$\xi = |t|^{-\nu} g(h/|t|^{\Delta})$$

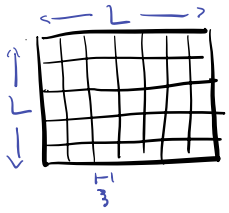
$$\xi(t=0,h) = h^{\nu_h} \to \nu_h = \nu/\Delta$$

2) Near criticality $\xi$ defines the length scale of the system and is solely responsible for divergence behavior

$\xi \gg a$ $\quad$ think about thermodynamic quantities as integrals of correlation function

Under these assumptions, we make the following argument. $\ln Z(t,h)$ must be dimensionless and extensive. This implies that the log of the partition function must take a form like

Consider a system with Length $L$. Near criticality behavior of the system is dictated by regions of size $\xi^d$




Each of these volumes acts as an independent rv contributing to $\ln Z$

$$\ln Z = \left(\frac{L}{\xi}\right)^d g_s + \cdots \left(\frac{L}{a}\right)^d * g_a$$

dimensionless — dimensionless

Extensive dependence on $L^d$

Singular contributions are from $(L/\xi)^d$

$$f_{sing}(t,h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d} \sim |t|^{d\nu} g_f\left(h/|t|^{\Delta}\right)$$

Assumption 1

Under these assumptions we recover Josephson's identity

$$2-\alpha = d\nu$$

Recall the two-point correlation function

$$G(x) \sim \frac{e^{-x/\xi}}{|x|^{d-2+\eta}}$$ ← fluctuation correction

Near criticality, $\xi \to \infty$ which implies that the correlation function scales as $1/|x|^{d-2+\eta}$

We can write our response functions as integrations of our correlation function

$$\chi \sim \int d^d x\, G^c_{mm}(x) \sim \int^{\xi} \frac{d^d x}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} \quad \Rightarrow \gamma = (2-\eta)\nu$$

correlations exponentially suppressed beyond $\xi$

$$C \sim \int d^d x\, G^c_{EE}(x) \sim \int^{\xi} \frac{d^d x}{|x|^{d-2+\eta'}} \sim \xi^{2-\eta'} \sim |t|^{-\nu(2-\eta')} \quad \Rightarrow \alpha = (2-\eta')\nu$$

Scaling introduces an additional dilation symmetry

Statistical properties are preserved (up to multiple) for different scaled perspectives

# Renormalization Group

The success of scaling hypothesis arguments in recapitulating observed behavior indicates that the only relevant length scale near criticality is $\xi$

Kadanoff's renormalization group attempts to reduce correlated degrees of freedom for $|x| << \xi$

General Outline:

1) Coarse Grain: Change the scale of the coarse graining so $a \to ba$ for $b > 1$

$$m_i(x) = \frac{1}{b^d} \int d^d x \, m_i(x')$$

2) Rescale: Restore resolution by scaling lengths

$$x_{new} = \frac{x_{old}}{b}$$

3) Renormalize: Introducing a change of constraint factor

$$\vec{m}_{new}(x_{new}) = \frac{1}{Zb^d} \int d^d x' \, \vec{m}(x')$$

Under this construction the statistics of $\vec{m}_{new}$ should be similar to $\vec{m}_{old}$ since $x << \xi$

The critical Hamiltonian is invariant under scaling and renormalization

$$\mathcal{H} = H_b \quad @ \text{ criticality}$$

If we move away from criticality, Kadanoff postulates that the Hamiltonian is governed by $t_{new}$ and $h_{new}$

We can consequently consider the action of RG as a mapping $t_{new} = t_b(t_{old}, h_{old})$

$$h_{new} = h_b(t_{old}, h_{old})$$

These new parameters can be written as analytic functions of previous parameters

$$t_b(t,h) = A(b)\,t + B(b)\,h + \cdots$$

$$h_b(t,h) = C(b)\,t + D(b)\,h + \cdots$$

Our Hamiltonian is invariant under the symmetry

$$m(x) \to -m(x)$$

$$h \to -h$$

$$t \to t$$

which implies that $B = 0$ and $C = 0$

We can repeatedly apply RG so $t(b_1 b_2) \approx A(b_1)A(b_2)\,t \approx A(b_1 b_2)\,t$

$$\Rightarrow A(b) = b^{y_t}$$

$$D(b) = b^{y_h}$$

Intuitively RG is restructuring the terms and weights in the Hamiltonian. It follows that the partition function is unperturbed

$$Z = \int Dm\, W[m] = \int Dm'\, W'[m'] = Z'$$

Consequently, free energy of the initial and RG are related

$$V f(t,h) = V' f(t',h') \quad \leftarrow \ln Z = \ln Z'$$

$$f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

If we select $b = t^{-1/y_t}$

$$f(t,h) = t^{d/y_t} f(1, b^{-y_h/y_t} h) = t^{d/y_t} g_f\left(h/t^{y_h/y_t}\right)$$

This is the same form as our earlier scaling hypothesis

$$2 - \alpha = d/y_t \qquad \Delta = y_h/y_t$$

A similar line of reasoning can be applied to correlation length which is scaled by $b$ during RG

$$\xi(t,h) = b\,\xi(b^{y_t} t, b^{y_h} h) = t^{-1/y_t} \cdot \xi\left(1, h/t^{y_h/y_t}\right) \sim t^{-\nu}$$

$$\nu = 1/y_t$$

This also recovers the desired hyperscaling relationship $2 - \alpha = d\nu$

For magnetization,

$$m(t,h) = -\frac{\partial f}{\partial h} = -\frac{1}{V}\frac{\partial \ln Z}{\partial h} = -\frac{1}{b^d V'}\frac{\partial \ln Z'(t,h')}{b^{y_h}\,\partial h'} = b^{y_h - d}\, m(b^{y_t} t, b^{y_h} h)$$

(chain rule)

Selecting $b = t^{-1/y_t}$ again, $\beta = \dfrac{y_h - d}{y_t}$ and $\Delta = y_h/y_t$

Renormalization procedure produces a mapping from the parameter space of the original Hamiltonian to the scaled picture

$$S = (t, u, v, \ldots K, L, \ldots)$$

$$S' = (t', u', v', \ldots K', L', \ldots) \qquad t' = t_b(t, u, \ldots) \text{ etc.}$$

$$S' = \mathcal{R}_b S$$

($\mathcal{R}_b$: dilation map)

Statistically self-similar configurations are fixed points of this transformation

Correlation length is only preserved when it is $0$ or $\infty$ (since RG scales it by $b$)

Fixed points corresponding to $\xi^* = 0$ have independent fluctuations (complete disorder, infinite temp)

$\xi^* = \infty$ is a critical point (complete order, zero temp)

We can study fixed points by linearizing around them

$$S_\alpha^* + \delta S'_\alpha = S_\alpha^* + (R_b^L)_{\alpha\beta}\, \delta S_\beta + \dots$$

$$(R_b^L)_{\alpha\beta} = \left.\frac{\partial S'_\alpha}{\partial S_\beta}\right|_{S^*}$$

We can diagonalize $(R_b^L)$ to get eigenvectors $O_i$ and eigenvalues $\lambda(b)_i$

From the group property, our transformation matrices commute and can be jointly diagonalized

Therefore, $\lambda(b)_i\, \lambda(b')_i = \lambda(bb')_i$

$$\lambda(b)_i = b^{y_i}$$

Any Hamiltonian in the vicinity of the fixed point can be represented

$$S = S^* + \sum_i g_i O_i$$

$$S' = S^* + \sum_i g_i b^{y_i} O_i \quad \leftarrow \text{Renormalized Hamiltonian}$$

We classify directions around the fixed point based on the sign of $y_i$

$y_i > 0$, $g_i$ increases under scaling and $O_i$ is a relevant operator

$y_i < 0$, $g_i$ decreases under scaling and $O_i$ is an irrelevant operator

$y_i = 0$, $O_i$ is a marginal operator whose higher order terms are necessary to track behavior

The subspace spanned by irrelevant operators defines the basin of attraction for a fixed point $S^*$

Under RG, correlation length scales like

$$\xi(g_1, g_2, \dots) = b\, \xi(b^{y_1} g_1, b^{y_2} g_2, \dots)$$

In the basin of attraction, repeated scaling gives us $g_i^{(n)} \to 0$

So repeated scaling from a point, $g$, in the basin

$$\xi(g) = b^n \xi(g^{(n)}) \to b^n \xi(S^*) = \infty \quad \leftarrow \text{Assuming fixed point is critical}$$

Therefore $\xi(g) = \infty$ for all $g$ in the basin

Singularities in $\xi$ arise from the relevant operators

Universality class defines a set of systems whose microscopic details wash out to produce the same relevant operators. The singular behavior of these systems is identical since only the relevant operators contribute

So how do we actually implement the RG transformation?

# RG in the Gaussian Model

Only consider quadratic terms in the Landau - Ginzberg Expansion

$$Z = \int \mathcal{D}m(x) \exp\left\{ -\int d^d x \left[ \frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \dots - h \cdot m \right] \right\}$$

Since there is no $m^4$ term to enforce stability $t \geq 0$

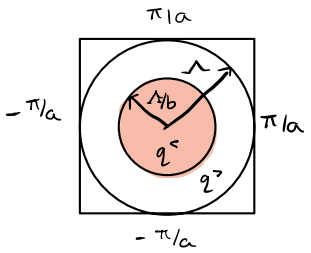
This system is readily analyzable in Fourier space

The allowed values of $q$ are discritized with spacing $\frac{2\pi}{L}$

The largest values of $q$ are limited by lattice spacing which defines the Brillouin zone

$$m(q) = \int d^d x \, e^{iqx} m(x)$$

$$m(x) = \int \frac{d^d q}{(2\pi)^d} e^{-iqx} m(q)$$




Allowed $q$ values from lattice structure

$$\beta H = \sum_q \left( \frac{t + Kq^2 + Lq^4 + \dots}{2V} \right) |m(q)|^2 - h \cdot m(q=0)$$

Skipping many steps (See Chapter 3)

$$Z = \exp\left[ \frac{Vh^2}{2t} \right] \prod_q \left( \frac{2\pi}{t + Kq^2 + Lq^4 + \dots} \right)^{n/2}$$

$n$ is the dimension of the field

The corresponding free energy

$$f(t,h) = -\frac{\ln Z}{V} = \frac{n}{2} \int_{BZ} \frac{d^d q}{(2\pi)^d} \ln\left( t + Kq^2 + Lq^4 + \dots \right) - \frac{h^2}{2t}$$

Contributions from the edge of the BZ are analytic

First we approximate the Brillouin zone by a hypersphere w radius $\Lambda \approx \pi/a$

$$f_{sing}(t,h) = \frac{n}{2} k_d \int_0^{\Lambda} dq \, q^{d-1} \ln\left( t + Kq^2 + Lq^4 + \dots \right) - \frac{h^2}{2t}$$

$$k_d = \frac{S_d}{(2\pi)^d} \leftarrow \text{d dimensional solid angle}$$

Exploiting spherical symmetry to simplify free energy

Rescale $q$ by a factor of $\sqrt{t/K}$

$$f_{sing}(t,h) = \frac{n}{2} k_d \left( \frac{t}{K} \right)^{d/2} \int_0^{\Lambda \cdot \sqrt{K}/\sqrt{t}} dx \, x^{d-1} \cdot \left[ \ln t + \ln\left( 1 + x^2 + \frac{2tx^4}{K^2} + \dots \right) \right] - \frac{h^2}{2t}$$

(contributes for large $q$)

$\ln(1+x^2)$ eventually $A$

Leading Singular Dependence

$$f_{sing}(t,h) \sim -t^{d/2} \left[ A + \frac{h^2}{2t^{1+d/2}} \right] = t^{2-\alpha} g_f\left( h/t^4 \right)$$

Basically finding singular $t$ behavior near $q = 0$

Now, let's consider the same problem from RG

$$Z \sim \int D\tilde{m}(q) \exp\left[-\int_0^{\Lambda} \frac{d^d q}{(2\pi)^d}\left(\frac{t + Kq^2 + Lq^4 + \cdots}{2}\right)|m(q)|^2 + h\cdot m(0)\right]$$

## Step 1: Coarse Grain

Removing fluctuation at scales $a < x < ba$ is equivalent to removing fourier modes

$$\Lambda/b < q < \Lambda$$

$$Z = \int Dm(q^<) \int D\sigma(q^>)\, e^{-\beta H[m,\sigma]}$$

Modes decouple in gaussian model

$$Z \sim \exp\left[-\frac{n}{2} V \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ln(t + Kq^2 + Lq^4 + \cdots)\right] \times \int Dm(q^<) \times \exp\left[-\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d}\left(\frac{t + Kq^2 + Lq^4 + \cdots}{2}\right)|m(q)|^2 + h\cdot m(0)\right]$$

## Step 2: Rescale

The integral for $m(q^<)$ is the same as the classic hamiltonian with the cutoff scaled to $\Lambda/b$

After rescaling space and momentum by $b$ $\quad x' = x/b \quad q' = q\cdot b$

$$Z = e^{-V \delta f_b(t)} \times \int D\tilde{m}(q') \exp\left[-\int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d} b^{-d} \times \left(\frac{t + Kb^{-2}q'^2 + Lb^{-4}q'^4 + \cdots}{2}\right)|m(q')|^2 + h\cdot|m(0)|\right]$$

## Step 3: Renormalize

Renormalize $m'(x) = m(x)/z$ or $m'(q') = m(q')/z$

$$Z = e^{-V \delta f_b(t)} \times \int D\, m'(q') \times \exp\left[-\int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d} b^{-d} z^2 \times \left(\frac{t + Kb^{-2}q'^2 + Lb^{-4}q'^4 + \cdots}{2}\right)|m'(q')|^2 + zh\cdot m'(0)\right]$$

From here we can read off

$$t' = b^{-d} z^2 t$$
$$h' = zh$$
$$K' = b^{-d-2} z^2 K$$
$$\vdots$$

To ensure fluctuations are scale invariant (Dont want fluctuations to change across RG)

$$z = b^{(d+2)/2} \quad \Rightarrow \quad K = K'$$

The relevant directions are thus

$$t' = b^2 t \qquad y_t = 2$$
$$h' = b^{d/2+1} h \longrightarrow y_h = 1 + d/2$$

The fixed point that satisfies the RG sets

$$t^* = h^* = L^* = \cdots 0$$

$$\beta\mathcal{H}^* = \frac{K}{2}\int_a^d dx\ (\nabla m)^2$$

For a generic $m^n(x)$ perturbation to the RG fixed point

$$\beta\mathcal{H}^* + u_n\int d^dx\ m^n \longrightarrow \beta\mathcal{H}^* + u_n b^d y^n \int d^dx\ m'^n$$

This suggests that

$$u_n' = u_n \cdot b^d\, b^{n\left(\frac{d-2}{2}\right)} \longrightarrow y_n = n - d\left(\frac{n}{2}-1\right)$$

4-d $\rightarrow$ $y_4$ is the next most important operator but it is irrelevant for $d>4$

$y_6$ is only relevant for $d<3$

Majority of operators are irrelevant at the gaussian fixed point for $d>2$

## 5. Perturbative Renormalization Group

Can we treat the Landau-Ginzberg Hamiltonian as a perturbation to the gaussian model?

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} = \int d^dx\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right]$$

$$+ u\int d^dx\ m^4 + \cdots$$

The gaussian model has a clean decoupling of fourier modes while the perturbative term introduces mixing

$$\mathcal{U} = u\int d^dx\ m^4 + \cdots$$

$$= u\int d^dx \int \frac{d^dq_1 d^dq_2 d^dq_3 d^dq_4}{(2\pi)^{4d}}\ e^{-ix(q_1+q_2+q_3+q_4)}\ m_\alpha(q_1)\, m_\alpha(q_2)\, m_\beta(q_3)\, m_\beta(q_4)$$

Summation over $\alpha$ and $\beta$ is implicit (components)

$$(m^2)^2 = (m_\alpha m_\alpha)(m_\beta m_\beta)$$

Fourier identity sets the condition $q_1+q_2+q_3+q_4 = 0$

$$\mathcal{U} = u\int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}}\ m_\alpha(q_1)\, m_\alpha(q_2)\, m_\beta(q_3)\, m_\beta(-q_1-q_2-q_3)$$

Wick's Theorem is useful to remember here

$$\left\langle \prod_{i=1}^{\ell} m_i \right\rangle = \begin{cases} 0 & \ell \text{ odd} \\ \text{sum pairwise contractions} & \ell \text{ even} \end{cases}$$

We can compute the expectation of an operator perturbatively as follows

$$\langle O \rangle = \frac{\int D\vec{m}\, O e^{-\beta(H_0 - U)}}{\int D\vec{m}\, e^{-\beta(H_0-U)}} = \frac{\int D\vec{m}\, O e^{-\beta H}\left[1 - U + U^2/2 - \cdots\right]}{\int D\vec{m}\, e^{-\beta H}\left[1 - U + U^2/2 - \cdots\right]}$$

$$= \frac{\langle O\rangle_0 - \langle UO\rangle_0 + \frac{1}{2}\langle OU^2\rangle_0 - \cdots}{1 - \langle U\rangle_0 + \langle U^2\rangle_0/2 - \cdots}$$

treating denominator like a power series

$$= \left[\langle O\rangle_0 - \langle OU\rangle_0 + \tfrac{1}{2}\langle OU^2\rangle_0 - \cdots\right]\left[1 + \langle U\rangle_0 + \langle U\rangle_0^2 - \tfrac{1}{2}\langle U^2\rangle_0 - \cdots\right]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3$$

$$= \langle O\rangle_0 - \left(\langle OU\rangle_0 - \langle O\rangle_0\langle U\rangle_0\right) + \frac{1}{2}\left(\langle OU^2\rangle_0 - 2\langle OU\rangle_0\langle U\rangle_0\right) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle OU^n\rangle_0^c$$

↖ cumulant

Consider the two-point correlation function of the LG model

$$U = u \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\, m_\alpha(q_1) m_\alpha(q_2) m_\beta(q_3) m_\beta(-q_1-q_2-q_3) + \cdots$$

$$\langle m_\alpha(q) m_\beta(q')\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle m_\alpha(q) m_\beta(q') U^n\rangle_0^c$$

1st order Cumulant expansion ↳

$$= \langle m_\alpha(q) m_\beta(q')\rangle_0 - u \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \times \Big[\langle m_\alpha(q) m_\beta(q') m_i(q_1) m_i(q_2) m_j(q_3) m_j(-q_1-q_2-q_3)\rangle_0 - \langle m_\alpha(q) m_\beta(q')\rangle_0 \langle m_i(q_1) m_i(q_2) m_j(q_3) m_j(-q_1-q_2-q_3)\rangle_0\Big]$$

$$+ O(u^2)$$

Now we need to compute $\langle OU\rangle_0$ which has $\binom{6}{2} = 15$ pairwise contractions

Consider casework:

1) Contractions between $\alpha, \beta$ and the remaining 4 m just cancel $\langle O\rangle_0\langle U\rangle_0$

Only surviving terms connect $O$ and $U$ (think pairings in wicks theorem)

This accounts for $\binom{4}{2} = 3$ pairings of the possible 15

2) $m_\alpha \leftrightarrow m_i$ ×2 $\quad m_\beta \leftrightarrow m_i \quad m_j \leftrightarrow m_j$ ×2 (Both $i,j$)

4 possible ways

$$\langle m_\alpha(q)\, m_i(q_1)\rangle_0 \langle m_\beta(q')\, m_i(q_2)\rangle_0 \langle m_j(q_3)\, m_j(-q_1-q_2-q_3)\rangle_0$$

$$= \frac{\delta_{\alpha i}\,\delta_{\beta i}\,\delta_{jj}\,(2\pi)^{3d}\,\delta^d(q+q_1)\,\delta^d(q'+q_2)\,\delta^d(q_1+q_2)}{(t+kq)^2\,(t+kq')^2\,(t+kq_3)}$$

$q_1$ and $q_2$ are coupled to $q$ and $q'$

Summing over all $i,j$ and integrating over appropriate $q_1, q', q_3$

$$-4u\,\frac{n\,\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d(q+q')}{(t+kq^2)^2}\int \frac{d^dq_3}{(2\pi)^d}\,\frac{1}{t+kq_3^2}$$

3) $m_\alpha \leftrightarrow m_i$ ×2 $\quad m_\beta \leftrightarrow m_j$ ×2 $\quad m_i \leftrightarrow m_j$ ×2 for $i,j$ invariance

$$\langle m_\alpha(q)\, m_i(q_1)\rangle_0 \langle m_\beta(q')\, m_j(q_3)\rangle_0 \langle m_i(q_2)\, m_j(-q_1-q_2-q_3)\rangle_0$$

$$= \frac{\delta_{\alpha i}\,\delta_{\beta j}\,\delta_{ij}\,(2\pi)^{3d}\,\delta(q+q_1)\,\delta(q'+q_3)\,\delta(q_1+q_3)}{(t+kq^2)\,(t+kq'^2)\,(t+kq_2^2)}$$

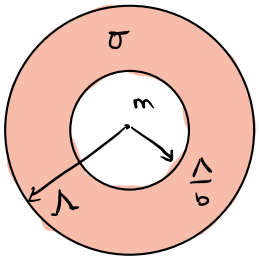
Summing and integrating appropriately

$$= -\,8u\,\frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta(q+q')}{(t+kq^2)^2}\int \frac{d^dq_2}{(2\pi)^d}\,\frac{1}{t+kq_2^2}$$

$$\langle m_\alpha(q)\, m_\beta(q')\rangle = \frac{\delta_{\alpha\beta}(2\pi)^d\,\delta(q+q')}{(t+kq^2)^2}\left[1 - \frac{4u(n+2)}{t+kq^2}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{t+Kk^2} + \mathcal{O}(u^2)\right]$$

# Perturbative RG (first Order)

## 1) Coarse grain




$$m(q) = \begin{cases} \tilde{m}(q) & \text{for } 0 < q < \Lambda/b \\ \sigma(q) & \text{for } \Lambda/b < q < \Lambda \end{cases}$$

$$Z = \int D\tilde{m}(q)\, D\sigma(q)\, \exp\left\{ -\int_0^{\Lambda} \frac{d^d q}{(2\pi)^d} \left( \frac{t + kq^2}{2} \right) \left( |\tilde{m}(q)|^2 + \sigma(q)^2 \right) - U[\tilde{m}(q), \sigma(q)] \right\}$$

We then integrate out $\sigma(q)$ to eliminate short distance detail

$$Z = \int Dm(q) \exp\left\{ -\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left( \frac{t + kq^2}{2} \right) |\tilde{m}(q)|^2 \right\} \times \exp\left\{ \frac{-nV}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ln\left(t + kq^2\right) \right\} \left\langle e^{-U[m,\sigma]} \right\rangle_\sigma$$

log determinant from gaussian RV

$$\langle O \rangle_\sigma = \int \frac{D\sigma(q)}{Z_\sigma} O \exp\left\{ -\int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \left( \frac{t + kq^2}{2} \right) |\sigma(q)|^2 \right\}$$

$$Z_\sigma = \int D\sigma(q) \exp\{-\beta H_0[\sigma]\}$$

The associated Hamiltonian is

$$\beta H[\tilde{m}] = V f_b^0 + \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left( \frac{t + kq^2}{2} \right) |\tilde{m}(q)|^2 - \ln \left\langle e^{-U[m,\sigma]} \right\rangle_\sigma$$

We can appropriately expand

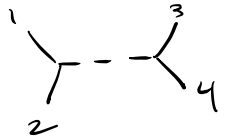
$$\ln \left\langle e^{-U} \right\rangle_\sigma = -\langle U \rangle_\sigma + \frac{1}{2} \left( \langle U^2 \rangle_\sigma - \langle U \rangle_\sigma^2 \right) + \frac{(-1)^l}{l!} \times l^{\text{th}} \text{ cumulant of } U$$

To the first order

$$\langle U[m,\sigma]\rangle_\sigma = u\int \frac{d^d q_1\, d^d q_2\, d^d q_3\, d^d q_4}{(2\pi)^{4d}}\,(2\pi)^d\,\delta^d(q_1+q_2+q_3+q_4)$$

$$\langle [m(q_1)+\sigma(q_1)]\,[m(q_2)+\sigma(q_2)]\,[m(q_3)+\sigma(q_3)]\,[m(q_4)+\sigma(q_4)]\rangle_\sigma$$

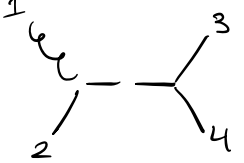
[1] $\langle \tilde{m}(q_1)\,\tilde{m}(q_2)\,\tilde{m}(q_3)\,\tilde{m}(q_4)\rangle_\sigma$




$U[\tilde{m}]$

[2] $\langle \sigma(q_1)\,\tilde{m}(q_2)\,\tilde{m}(q_3)\,\tilde{m}(q_4)\rangle_\sigma$

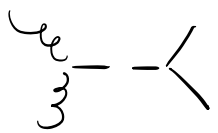
×4




0

[3] $\langle \sigma(q_1)\,\sigma(q_2)\,\tilde{m}(q_3)\,\tilde{m}(q_4)\rangle_\sigma$

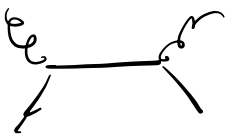
×2



$(\sigma\cdot\sigma)(m\cdot m)$

[4] $\langle \sigma(q_1)\,m(q_2)\,\sigma(q_3)\,m(q_4)\rangle$

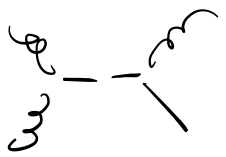
×2 ×2 ×4



$(\sigma\cdot m)(\sigma\cdot m)$

[5] $\langle \sigma(q_1)\,\sigma(q_2)\,\sigma(q_3)\,\tilde{m}(q_4)\rangle_\sigma$

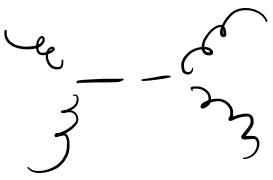
×4



0

[6] $\langle \sigma(q_1)\,\sigma(q_2)\,\sigma(q_3)\,\sigma(q_4)\rangle$

×1



Term [3]

$$-2u\int \frac{d^d q_1 \cdots d^d q_4}{(2\pi)^{4d}}\,(2\pi)^d\,\delta^d(q_1\cdots q_4)\,\frac{\delta_{jj}\,(2\pi)^d\,\delta(q_1+q_2)}{t+Kq_1^2}\,m(q_3)\,m(q_4)$$

$$= -2nu\int_0^{\Lambda/b}\frac{d^d q}{(2\pi)^d}\,|m(q)|^2\int_{\Lambda/b}^{\Lambda}\frac{d^d k}{(2\pi)^d}\,\frac{1}{t+Kk^2}$$

Following a similar procedure,

$$\mathcal{B}\tilde{\mathcal{H}}[\tilde{m}] = V\left(\delta f_b^0 + u\,\delta f_b^1\right) + \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d}\left(\frac{\tilde{t} + kq^2}{z}\right)|\tilde{m}(q)|^2$$

$$+ u \int_0^{\Lambda/b} \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\, m(q_1)m(q_2)m(q_3)\cdot m(-q_1-q_2-q_3)$$

$$\hat{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d}\,\frac{1}{t+kk^2}$$

2) Rescale $\quad q = b^{-1}q'$

3) Renormalize $\quad m = zm'$

$$(\mathcal{B}H)'[m'] = V\left(\delta f_b^0 + u\,\delta f_b^1\right) + \int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d}\, b^{-d} z^2 \left(\frac{\hat{t} + kb^{-2}q^2}{z}\right)|m(q')|^2$$

$$+ u z^4 b^{-3d} \int_0^{\Lambda} \frac{d^d q_1' d^d q_2' d^d q_3'}{(2\pi)^{3d}}\, m'(q_1')\cdot m'(q_2)\, m'(q_3)\, m'(-q_1-q_2-q_3)$$

The scaling relations we get from RG

$$t' = b^{-d} z^2 \hat{t} \qquad k' = b^{-d-2} z^2 k \qquad u' = b^{-3d} z^4 u$$

If we set $z = b^{1+d/2}$ we get $k' = k$ and a fixed point at $t^* = u^* = 0$

$$t_b' = b^2\left[t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d}\,\frac{1}{t+kk^2}\right]$$

$$u_b' = b^{4-d} u$$

It is common to consider infinitesimal RG $b = e^{\delta l}$

$$t'_b \equiv t(b) = t(1+\delta l) = t + \delta l \frac{dt}{dl} + O(\delta l^2)$$

For infinitesimal $b$ we approximate the integral $\int_{\Lambda/b}^{\Lambda}$ with $k = \Lambda$ and still $dl = \delta\Lambda$

$$t'_b = b^2 \left[ t + 4u(n+2) \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{t + k\Lambda^d} \delta l \right]$$

↓ expand $(1+\delta l)^2$ to first order

$$= (1+2\delta l) \left[ t + 4u(n+2) \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{t+k\Lambda^d} \delta l \right]$$

$$u'_b \equiv u(b) = u + \delta l \frac{du}{dl}$$

$$= (1 + (4-d)\delta l)\, u$$

To first order, we get differential equations

$$\frac{dt}{dl} = 2t + \frac{4u(n+2) k_d \Lambda^d}{t + k\Lambda^2}$$

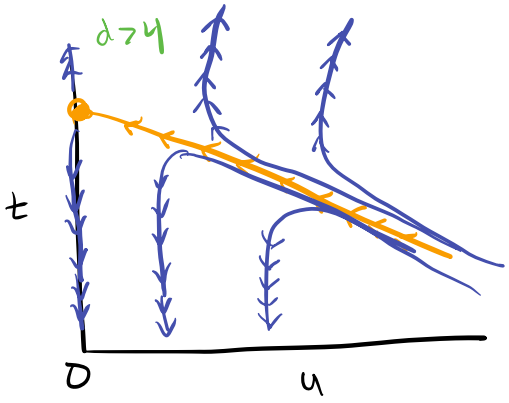
We can now linearize these recursion relations

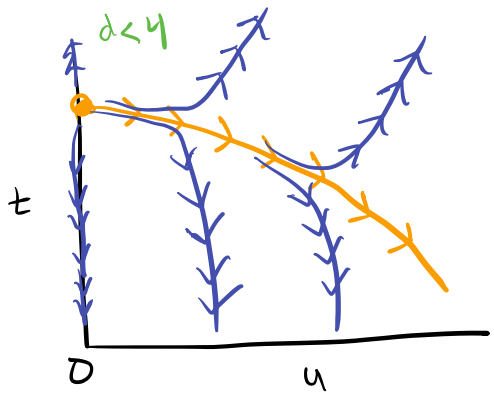
$$\frac{d}{dl} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & \frac{4(n+2) k_d \Lambda^{d-2}}{15} \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix}$$

We can read relevant operators as eigenvalues of this matrix

$y_t = 2$ $\quad y_u = 4-d$

direction changed



Correctly describes phase transition for $d > 4$ since only one unstable direction

Incorrect description for $d < 4$ since there are two relevant directions

We could guess the existence of second order corrections

$t$ : $-Au^2$

$u$ : $+Bu^2$

which creates a new fixed point at

$u^* = (4-d)/B$ for $d < 4$

For perturbation theory to be accurate we need $u$ to be small so we can only explore the vicinity of $\epsilon = 4-d$

## Perturbative RG (Second Order)

$$\beta\tilde{\mathcal{H}}[\vec{m}] = V s f_b^0 + \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left( \frac{t + kq^2}{2} \right) |m(q)|^2$$

$$+ \langle u \rangle_0 - \frac{1}{2} \left( \langle u^2 \rangle_0 - \langle u \rangle_0^2 \right) + \mathcal{O}(u^3)$$

Now need to consider 36 terms in expansion

Many of the terms cancel

Skipping section but mechanically the same as first-order

Revisit $\epsilon$-expansion

# 6. Lattice Systems

Universality allows us to study simple lattice systems w/ interaction energies and recover the same macroscopic physics

These models allow for more tractable analysis and simulation

## Ising Model

Each lattice site $i$ takes spin $\sigma_i \in (-1, +1)$

Short-range interactions involve neighboring spins

$$\mathcal{H} = \sum_{\langle i,j \rangle} \hat{\mathcal{B}}(\sigma_i, \sigma_j)$$

$$\hat{\mathcal{B}}(\sigma,\sigma') = -g - \frac{h}{z}(\sigma+\sigma') - J\sigma\sigma' \quad \leftarrow \text{General Interaction}$$

$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}} = \sum_{\{\sigma_i\}} \exp\left[K \sum_{\langle i,j \rangle} \sigma_i\sigma_j + h\sum_i \sigma_i + g\right]$$

$z$ corresponds to the number of bonds per site

At $T=0$ ($h=0$) the system is degenerate between all spins up or down

At some critical $k_c = \frac{J}{k_B T_c}$ there is a phase transition into a disordered state

$h$ breaks this symmetry and removes the phase transition

## Om Model

Each lattice site is occupied by an $n$-component unit vector

$$S_i = (S_i^1, S_i^2, \dots S_i^n) \quad \text{s.t.} \quad \sum_{\alpha=1}^{n} (S_i^\alpha)^2 = 1$$

Nearest neighbor interactions generalize as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - h \cdot \sum_i S_i \quad \leftarrow \text{Consistent w/ Rotational Symmetry when } h=0$$

## Potts Model

Each lattice site is occupied by a $q$-valued spin $S_i = 1, 2, \dots q$

Neighboring interactions define the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{S_i S_j} - \hat{h} \sum_i \delta_{S_i, 1}$$

$h$ breaks permutation symmetry among the $q$ states

$q > 2$ defines a new universality class

## Spin $s$-models

The spin at each site takes $2s+1$ values

$$S_i = -s, -s+1, \dots s$$

Corresponding Hamiltonian

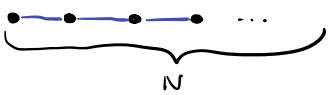
$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(J_1 S_i S_j + J_2 (S_i S_j)^2 + \dots J_{2s} (S_i S_j)^{2s}\right) - \hat{h} \sum_i S_i$$

Ising model is recovered when $s = 1/2$

$s=1$ is the Blume-Emery-Griffith model describing a mix of magnetic and non-magnetic components

These models can be studied through exact solutions (1D systems, 2D Ising Model), Position Space Renormalizations, Monte Carlo simulations, series expansions

# Transfer Matricies




Consider a linear chain of $N$ ising spins $\sigma_i = \pm 1$ with nearest neighbor coupling $K$ and magnetic field $h$

Assume periodic boundary conditions such that the two ends are coupled

$$-\beta\mathcal{H} = K(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \dots + \sigma_{N-1}\sigma_N + \sigma_N\sigma_1) + h\sum_{i=1}^{N}\sigma_i$$

$$Z = \sum_{\{\sigma_i\}} \exp\{-\beta\mathcal{H}\} = \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\prod_{i=1}^{N}\exp\left[K\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i+\sigma_{i+1})\right]$$

$N+1 = 1$

This resembles matrix mult over $\sigma_i$

$$\equiv \mathrm{tr}\left[\langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle\dots\langle\sigma_N|T|\sigma_1\rangle\right]$$

$$= \mathrm{tr}\left[T^N\right]$$

Where the $2\times2$ Transfer Matrix is defined

$$\langle\sigma_i|T|\sigma_j\rangle = \exp\left[K\sigma_i\sigma_j + \frac{h}{2}(\sigma_i+\sigma_j)\right]$$

$$T = \begin{matrix}+1\\-1\end{matrix}\overset{\begin{matrix}+1 & -1\end{matrix}}{\begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h}\end{bmatrix}}$$

If we diagonalize $T$ we can write the tr as

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N\left[1 + \left(\lambda_-/\lambda_+\right)^N\right] \approx \lambda_+^N$$

$$T^N = \begin{pmatrix}\lambda_+^N & 0\\ 0 & \lambda_-^N\end{pmatrix}$$

Assuming that $\lambda_+ > \lambda_-$

In the limit $N \to \infty$

$$\beta f = -\frac{\ln Z}{N} = -\ln\lambda_+$$

So, now we consider the eigenvalues of $T$

$$\lambda_\pm = e^K\cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}}$$

Now, consider the average spin at site $i$

$$\langle\sigma_i\rangle = \frac{1}{Z}\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\sigma_i\prod_{i=1}^{N}\exp\left(K\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i+\sigma_{i+1})\right)$$

$$= \frac{1}{Z}\mathrm{tr}\left[\langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle\dots\langle\sigma_{i-1}|T|\sigma_i\rangle\sigma_i\langle\sigma_i|T|\sigma_{i+1}\rangle+\dots\langle\sigma_N|T|\sigma_1\rangle\right]$$

$$\sum_{\{\sigma_i\}}\langle\sigma_{i-1}|T|\sigma_i\rangle\sigma_i\langle\sigma_i|T|\sigma_{i+1}\rangle = \langle\sigma_{i-1}|T\hat{\sigma}_z T|\sigma_{i+1}\rangle$$

Weighted Matrix Contraction

$$\hat{\sigma}_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \leftarrow \text{Pauli Matrix}$$

$$= \frac{1}{Z}\mathrm{tr}\left[T^{i-1}\hat{\sigma}_z T^{N-i+1}\right]$$

$$= \frac{1}{Z}\mathrm{tr}\left[T^N\hat{\sigma}_z\right]$$

If we diagonalize $T$ when $h = 0$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \leftarrow \text{Rotation Matrix}$$

$$\langle \sigma_i \rangle = \frac{1}{Z} \operatorname{tr} \left[ \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{\sigma}_z \text{ rotated pauli matrix}} \right] = \frac{1}{Z} \operatorname{tr} \begin{pmatrix} 0 & \lambda_+^N \\ \lambda_-^N & 0 \end{pmatrix} = 0$$

Magnetization vanishes with no external field by symmetry

We can similarly compute the two-spin correlation function

$$\langle \sigma_i \sigma_{i+r} \rangle = \frac{1}{Z} \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \sigma_i \sigma_{i+r} \prod_{j=1}^{N} \exp\left( K \sigma_j \sigma_{j+1} + \frac{h}{2} (\sigma_j + \sigma_{j+1}) \right)$$

$$= \frac{1}{Z} \operatorname{tr} \left[ T^{i-1} \hat{\sigma}_z T^r \hat{\sigma}_z T^{N-i-r+1} \right]$$

Can I change $\sigma_i$ and $\sigma_{i+r}$ like the potts model or something

$$= \frac{1}{Z} \operatorname{tr} \left[ \hat{\sigma}_z T^r \hat{\sigma}_z T^{N-r} \right]$$

Once again diagonalizing

$$\langle \sigma_i \sigma_{i+r} \rangle = \frac{1}{Z} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_+^r & 0 \\ 0 & \lambda_-^r \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r} \end{pmatrix} \right]$$

$$= \frac{1}{Z} \operatorname{tr} \begin{pmatrix} \lambda_+^{N-r} \lambda_-^r & 0 \\ 0 & \lambda_-^{N-r} \lambda_+^r \end{pmatrix} = \frac{\lambda_+^{N-r} \lambda_-^r + \lambda_-^{N-r} \lambda_+^r}{\lambda_+^N + \lambda_-^N}$$

When $N \gg r$,

$$\langle \sigma_i \sigma_{i+r} \rangle \propto \left( \frac{\lambda_-}{\lambda_+} \right)^r = e^{-r/\xi}$$

$$\xi = \left[ \ln \left( \frac{\lambda_+}{\lambda_-} \right) \right]^{-1} = \frac{1}{\ln \tanh K}$$

This solution generalizes for spins $\{s_i\}$

$$Z = \sum_{\{s_i\}} \exp \left[ \sum_{i=1}^{N} \mathcal{B}(s_i, s_{i+1}) \right] = \sum_{\{s_i\}} \prod_{i=1}^{N} e^{\mathcal{B}(s_i, s_{i+1})}$$

$$\langle s_i | T | s_{i+1} \rangle = e^{\mathcal{B}(s_i, s_{i+1})}$$

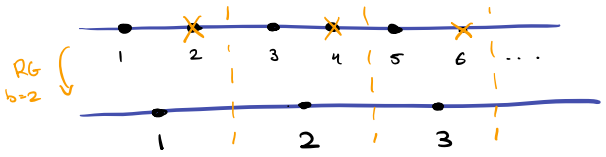
When there are periodic boundary conditions we consider the trace

$$Z = \operatorname{tr} [T^N] \propto \lambda_{max}^N$$

Singularities only exist when $T = 0$ since eigenvalues and ratios are well-behaved for finite-matrix w/ pos elements

To apply RG to the one dimensional system we must define a transformation to rescale the system

The simplest way is to consider decimation and ignore even indexed spins ($b=2$)




$$Z = \sum_{\{\sigma_i\}}^{N} \exp\left\{ \sum_{i=1}^{N} B(\sigma_i, \sigma_{i+1}) \right\} = \sum_{\{\sigma_i'\}}^{N/2} \sum_{\{s_i\}}^{N/2} \exp\left[ \sum_{i=1}^{N/2} \left[ B(\sigma_i', s_i) + B(s_i, \sigma_{i+1}') \right] \right]$$

$s_i$ denotes decimated spins

Summing over $s_i$

$$e^{-\beta \mathcal{H}'[\sigma_i']} = \prod_{i=1}^{N/2} \left[ \sum_{\{s_i\}}^{N/2} \exp\left\{ B(\sigma_i', s_i) + B(s_{i+1}, \sigma_{i+1}') \right\} \right] = \exp\left[ \sum_{i=1}^{N/2} B'(\sigma_i', \sigma_{i+1}') \right]$$

where $\quad B(\sigma_1, \sigma_2) = g + \frac{h}{2}(\sigma_1 + \sigma_2) + K \sigma_1 \sigma_2$

$$B'(\sigma_1, \sigma_2) = g' + \frac{h'}{2}(\sigma_1' + \sigma_2') + K' \sigma_1' \sigma_2'$$

This implies that in the renormalized interaction

$$R(\sigma_1', \sigma_2') = \exp\left[ g' + \frac{h'}{2}(\sigma_1' + \sigma_2') + K' \sigma_1' \sigma_2' \right]$$

$$= \sum_{s_i = \pm 1} \exp\left[ 2g + h s_1 + K s_1 (\sigma_1' + \sigma_2') + \frac{h}{2}(\sigma_1' + \sigma_2') \right]$$

Consider

$$x = e^{K} \qquad y = e^{h} \qquad z = e^{g}$$

$$x' = e^{K'} \qquad y' = e^{h'} \qquad z' = e^{g'}$$

For each configuration of the board, we get

$$R(+,+) = z' y' x' = z^2 y^2 x^2 + z^2 x^{-2} = z^2 y \left( x^2 y + x^{-2} y^{-1} \right)$$

$$R(-,-) = x' y'^{-1} z' = z^2 y^{-1} \left( x^{-2} y + x^2 y^{-1} \right)$$

$$R(-,+) = x'^{-1} z' = z^2 (y + y^{-1})$$

$$R(+,-1) = x'^{-1} z' = z^2 (y + y^{-1})$$

These result in the following recursion relation (skipping algebra)

$$g' = 2g + \delta g(K, h)$$

$$h' = h + \delta h(K, h)$$

$$K' = K'(K, h)$$

When $h=0$ we get $y=1 \Rightarrow y'=1$

$$e^{4K'} = \left(\frac{e^{2K}+e^{-2K}}{2}\right)^{-2} \Rightarrow K' = \frac{1}{2}\ln\cosh K$$

This implies that $K^*=0$ (infinite temperature) is a fixed point

For small $K$ we get $K' = \ln\left(\frac{1+4K^2}{2}\right) \sim K^2$ which implies that this is stable

Similarly, $K^*=\infty$ (zero temp) is another fixed point although this is unstable

$$K' \approx \ln\left(e^{2K}/2\right) \approx K - \frac{\ln 2}{2}$$




Finite temperature model is always disordered at large enough length scales

Correlations diverge at $K \to \infty$, $x \to \infty$. Linearize around this fixed point

From previously $x'^4 = \dfrac{(x^2y + x^{-2}y^{-1})(x^{-2}y + x^2y^{-1})}{(y+y^{-1})^2}$

↓ gives us scaling fields

As $x \to \infty$, $x'^4 \approx \dfrac{x^4}{(y+y^{-1})^2}$

Near $h=0$, $y+y^{-1} \approx 2$

$$x'^4 \approx \frac{x^4}{4}$$

Similarly,

$$y'^2 = y^2 \frac{x^2y + x^{-2}y^{-1}}{x^{-2}y + x^2y^{-1}}$$

$$\approx y^4$$

So we get

$$x'^4 \approx x^4/4 \qquad y'^2 \approx y^4 \Rightarrow$$

$$e^{K'} \approx e^K \cdot 1/\sqrt{2} \quad \text{or} \quad e^{-K'} = \sqrt{2}\,e^{-K}$$

$$h' = 2h$$

If we apply RG at $\xi = \infty$

$$\xi(e^{-K}, h) = 2\xi(\sqrt{2}e^{-K}, 2h)$$

↑ scaling fields

$$= 2^\ell \xi(2^{\ell/2}e^{-K}, 2^\ell h)$$

Selecting $\ell$ s.t. $2^{\ell/2}e^{-K} = 1$ gives us the scaling relation

$$\xi(e^{-K}, h) = e^{2K} g_\xi(he^{2K})$$

Hyper scaling argues that

$$f_{sing} \sim \xi^{-d} = \xi^{-1} = e^{-2K} g_f(he^{2K})$$

Susceptibility follows

$$\chi(K) \sim \frac{\partial^2 f}{\partial^2 h}\Big|_{h=0} \sim e^{2K}$$

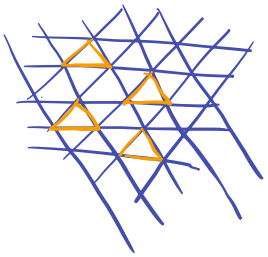
Since $\chi \sim \xi^{2-\eta}$ we get $\eta = 1$

Decimation procedure does not generalize cleanly to higher dimensions

Removing one spin couples its bond neighbors to produce 4-point interactions

## Niemeijer-van Leeuwen Cumulant Approximation

Treat the 2-D lattice as a grid of triangles w/ intercell interactions treated as perturbations



$$\sigma'_\alpha = \text{sign}\left[\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3\right] \leftarrow \text{majority rule}$$

Decompose RG hamiltonian

$$\mathcal{BH} = \mathcal{BH}_0 + U$$

$$-\mathcal{BH}_0 = K \sum_\alpha \left(\sigma_\alpha^1 \sigma_\alpha^2 + \sigma_\alpha^2 \sigma_\alpha^3 + \sigma_\alpha^1 \sigma_\alpha^3\right)$$

Exact intracell interactions

$$-U = K \sum_{\langle \alpha,\beta \rangle} \left(\sigma_\beta^{(1)} \sigma_\alpha^{(2)} + \sigma_\beta^{(1)} \sigma_\alpha^{(3)}\right)$$

Neighboring cells w/ 2 bonds

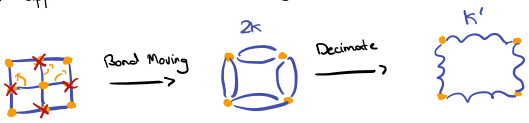
Perturbative Hamiltonian via Cumulant Expansion

$$\mathcal{BH}'[\sigma'_\alpha] = -\ln Z_0[\sigma'_\alpha] + \langle U \rangle_0 - \frac{1}{2}\left(\langle U^2 \rangle_0 - \langle U \rangle_0^2\right) + \mathcal{O}(U^3)$$

Skipping Computations

## Migdal-Kadanoff Bond Moving Approximation

This approach considers moving bonds during decimation in a 2D lattice




$$K' = \frac{1}{2}\ln\cosh(2 \cdot 2K)$$

There are 3 fixed points under this rescaling

a) $K^* = 0$, when $K \ll 1$, $K' \approx \ln\frac{(1+8K^2)}{2} \approx K^2$ → stable fixed point

b) $K^* = \infty$, when $K \gg 1$, $K' \approx \ln\left(e^{4K}/2\right) \approx 2K$ → Also stable fixed point

c) $$e^{2K^*} = \frac{e^{4K^*} + e^{-4K^*}}{2} \Rightarrow K^* \approx 0.305$$

We can once again linearize around the fixed point

$$b^{y_t} = \frac{\partial K'}{\partial K}\Big|_{K^*} = 2\tanh 4K^* \approx 1.6796$$

$$y_t = 0.747$$

In higher dimensions the band moving process generalizes with strength changing by $2^{d-1}$

$$K' = \frac{1}{2}\ln\cosh\left(2\times 2^{d-1}K\right)$$

$K^* = 0$ and $\infty$ remain stable

$$2^{y_t} = \left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{d-1}\tanh\left(2^d K^*\right)$$

Approximation gets worse for higher dimensions

## 7. Series Expansions

Another approach to studying Lattice models is through series expansions in which exact solutions are perturbed

### Low Temperature Expansions

Consider an ising model on a d-dimensional hypercube with the following Hamiltonian

$$-\beta\mathcal{H} = K\sum_{\langle i,j\rangle}\sigma_i\sigma_j$$

The ground state solution is ferromagnetic with identical spins

Consider low energy fluctuations around this state

1) Single Overturned Spin

   N possible sites with an energy cost of $2K\cdot 2d$

2) Dimer of negative spins

   $N\times d$ orientations with corresponding cost $2K\times(4d-2)$

We expand the partition function as follows:

$$Z = 2e^{Ndk}\left[1 + Ne^{-4dK} + dNe^{-4(2d-1)K} + \frac{N(N-2d-1)}{2}e^{-8dK} + \cdots\right]$$

$2e^{Ndk}$: $+$ or $-$; all possible spins. $Ne^{-4dK}$: single spin perturb. $dNe^{-4(2d-1)K}$: Dimer perturb. $\frac{N(N-2d-1)}{2}e^{-8dK}$: 2 disjoint single perturb

Use $\ln(1+x)\approx x-\frac{x^2}{2}+\cdots$

Free energy follows (Drop 2 in infinite limit)

$$-\beta f = \frac{\ln Z}{N} = dK + e^{-4dK} + de^{-4(2d-1)K} - \frac{(2d+1)}{2}e^{-8dK} + \cdots$$

Higher order terms of N are cancelled in free energy by $\frac{x^2}{2}$ term

Relates to extensive property of free energy so only connected terms remain

Can think of the rotations factorizing and not surviving the log

We can consequently compute energy and heat capacity

$$\frac{E}{N} = -\frac{\partial}{\partial\beta}\left(\frac{\ln Z}{N}\right) = -J\frac{\partial}{\partial K}\left(\frac{\ln Z}{N}\right)$$

$$= -J\left[d - 4de^{-4dK} - 4d(2d-1)e^{-4(2d-1)K} + 4d(2d+1)e^{-8dK} + \cdots\right]$$

$$\frac{C}{Nk_B} = \frac{1}{Nk_B}\frac{\partial E}{\partial T} = \frac{-K^2}{NJ}\frac{\partial E}{\partial K}$$

$$= K^2\left[16d^2e^{-4dK} + 16d(2d-1)^2e^{-4(2d-1)K} - 32d^2(2d+1)e^{-8dK} + \cdots\right]$$

### High-Temperature Expansions

In high temperature expansions we begin w/ independent spins and expand around powers of $\beta$

$$Z = \mathrm{tr}\left(e^{-\beta\mathcal{H}_0}\right) = \mathrm{tr}\left[1 - \beta\mathcal{H} + \frac{\beta^2\mathcal{H}^2}{2} - \cdots\right]$$

$$\frac{\ln Z}{N} = \frac{\ln Z_0}{N} - \beta\frac{\langle\mathcal{H}\rangle_0}{N} + \frac{\beta^2}{2}\frac{\langle\mathcal{H}^2\rangle_0 - \langle\mathcal{H}\rangle_0^2}{N} - \cdots$$

For an Ising model each spin take $\sigma_i \in \{-1, 1\}$

Its informative to write the Boltzmann weight for each bond as follows

$$e^{K\sigma_i\sigma_j} = \frac{e^{K} + e^{-K}}{2} + \frac{e^{K} - e^{-K}}{2}\sigma_i\sigma_j = \cosh K + \sinh K\ \sigma_i\sigma_j = \cosh K\left[1 + \tanh K\ \sigma_i\sigma_j\right]$$

This form is related to the functional basis on $\{-1, 1\}$

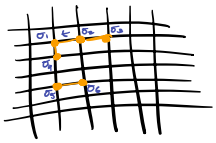
$t = \tanh K$ is a good high temperature expansion parameter ($\tanh K << 1$)

The partition function can then be written

$$Z = \sum_{\{\sigma_i\}} e^{K\sum_{\langle i,j\rangle}\sigma_i\sigma_j} = \sum_{\sigma_i}\prod_{\langle i,j\rangle}\cosh K\left[1 + t\sigma_i\sigma_j\right] = (\cosh K)^{\#\text{bonds}=N_b}\sum_{\{\sigma_i\}}\prod_{\langle i,j\rangle}\left[1 + t\sigma_i\sigma_j\right]$$

The product of $N_b$ bonds produces $2^{N_b}$ terms

Each term can be interpreted as the connection/ lack of connection on a lattice




$(t\sigma_1\sigma_2)(t\sigma_2\sigma_3)(t\sigma_4\sigma_5)(t\sigma_5\sigma_6)$

When we sum over $\{\sigma_i\}$ only terms w/ even number of bonds incoming survive

This implies that only closed loops survive on the lattice

The partition function appropriately simplifies as

$$Z = 2^N(\cosh K)^{N_b}\sum_{\text{closed graphs}} t^{\#\text{bonds}}$$

The smallest closed loop in a $d$-dimensional hypercube is a 4 bond square, the second 6 bond rectangle

$\frac{d(d-1)}{2}$ possible orientations $\qquad d(d-1)(2d-3)$

$$Z = 2^N(\cosh K)^{dN}\left[1 + \frac{d(d-1)N}{2}t^4 + d(d-1)(2d-3)t^6 + \cdots\right]$$

Essentially adding ordered perturbations to fully disordered system

$\ln(1+x) \sim x - \frac{x^2}{2}\cdots$

$$\beta f = \frac{\ln Z}{N} = \ln 2 + d\ln\cosh K + \frac{d(d-1)}{2}t^4 + \cdots$$

## Series Expansions for 1-D Ising Model

1) Open Chain Ising Model




$N-1$ bonds

No possible closed loops on this graph

$$Z = 2^N(\cosh K)^{N-1}$$

$$\frac{\ln Z}{N} = \ln 2 + \ln\cosh K - \frac{\ln\cosh K}{N}$$

Consider the correlation function between two lattice points $\sigma_n\sigma_m$

$$\langle\sigma_n\sigma_m\rangle = \frac{1}{Z}2^N(\cosh K)^{N-1}\sum_{\{\sigma_i\}}\sigma_n\sigma_m\prod_{\langle i,j\rangle}(1 + t\sigma_i\sigma_j)$$

Terms only survive if $\sigma_n$ and $\sigma_m$ have an odd number of bonds

The only stable chain (even bonds elsewhere) is achieved when $n$ and $m$ are connected

$$\langle \sigma_n \sigma_m \rangle = \frac{2^N (\cosh K)^{N-1}}{Z} t^{|m-n|} = t^{|m-n|} = e^{-|m-n|/\xi}$$

$$\xi = -\frac{1}{\ln \tanh K}$$

## 2) Closed Chain

Now there can be closed loops and N bonds

$$Z = 2^N (\cosh K)^N \left[1 + t^N\right]$$

↑ All bonds connected

$$\frac{\ln Z}{N} = \ln 2 \cosh K + \frac{\ln\left[1 + t^N\right]}{N}$$

free energy equals open chain in thermodynamic limit

Using same path counting arguments

$$\langle \sigma_n \sigma_m \rangle = \frac{2^N (\cosh K)^N}{Z} \left[ t^{|m-n|} + t^{N-|m-n|} \right] = \frac{\left( t^{|m-n|} + t^{N-|m-n|} \right)}{1 + t^N}$$

We can relate high temperature to low temperature expansions

(# bonds, ΔE)

Low Temp: $Z = e^{2NK}\left[1 + Ne^{-4 \times 2K} + 2Ne^{-6 \times 2K} + \cdots\right]$

$$= e^{2NK} \sum_{\text{island (-) spins}} e^{-2K \times \text{perimeter}}$$

High Temp: $Z = 2^N (\cosh K)^{2N}\left[1 + N \tanh K^4 + 2N \tanh K^6 + \cdots\right]$

$$= 2^N \cosh K^{2N} \sum_{\text{even bond count}} \tanh K^{\#\text{ graph length}}$$

The boundary of island spins in the low temp expansion satisfies the high temp loop conditions

This defines a 1-to-one correspondence between the expansions

Free energy can be written

$$\frac{\ln Z}{N} = 2K + g\left(e^{-2K}\right) = \ln 2 + 2 \ln \cosh K + g(\tanh K)$$

Low temp High Temp

This defines a duality condition

$$e^{-2\tilde{K}} \leftrightarrow \tanh K$$

$$\tilde{K} = D(K) = -\frac{1}{2} \ln \tanh K$$

The behavior of this dual function does the following

1) Low temp ⟷ High Temp
2) Dual interactions are symmetrically related

$$\sinh 2K \sinh 2\tilde{K} = 1$$

3) Singularities in $K_c$ must preserved in $\tilde{K}_c$

Self-duality condition

Duality arguments can be applied to 3D ising models

Low Temp: $Z = e^{3NK}\left[1 + Ne^{-2K\times 6} + 3Ne^{-2K\times 10} + \cdots\right]$ $\quad 1 + \square + \square\square + \cdots$

$= e^{3NK} \sum_{\substack{\text{island (-)} \\ \text{spins}}} e^{-2K \times \text{area of boundary}}$

High Temp: $Z = 2^N \cosh K^{3N}\left[1 + 3N \tanh K^4 + 18N \tanh K^6 + \cdots\right]$ $\quad 1 + 3N \square + 18N \square\!\square$

$= 2^N \cosh K^{3N} \sum_{\substack{\text{even bond} \\ \text{count}}} \tanh K^{\#\text{ lines}}$

The duals in this classical model don't align

Introduce a gauge invariant Hamiltonian dependent on plaquettes rather than lattice spins

$$\tilde{Z} = \sum_{\{\tilde{\sigma}_p^i = \pm 1\}} \prod_{\text{plaquettes}} \left(1 + \tanh \tilde{\sigma}_p^1 \tilde{\sigma}_p^2 \tilde{\sigma}_p^3 \sigma_p^4\right) \propto \sum_{\{\sigma_i\}} e^{\tilde{K} \sum_p \hat{\sigma}_{i3} \tilde{\sigma}_p^2 \tilde{\sigma}_p^3 \tilde{\sigma}_p^4}$$

This is considered a $\mathbb{Z}_2$ lattice gauge theory

Introduces local or gauge symmetry so changing spin changes adjacent plaquettes which each are computed from two charged spins



Plaquette Energy unchanged

Instead of considering phase transitions in the magnetization we consider long term order (correlation function)

Wilson loop is computed as a closed loop of S bonds

$$C_S = \langle \prod_{i=S} \hat{\sigma}_i \rangle$$

$C_S$ is gauge invariant since any spin change impacts two values in the loop

High Temp:

$$C_S = \frac{1}{Z} \sum_{\{\sigma_i\}} \prod \tilde{\sigma}_i \, e^{K \sum \hat{\sigma}_{ip}^1 \tilde{\sigma}_p^2 \tilde{\sigma}_p^3 \tilde{\sigma}_p^4}$$

$$= (\tanh \tilde{K})^{\text{Area of } S} \times \left[1 + \mathcal{O}(\tanh \tilde{K}^2)\right]$$

Area with boundary S + perturbations of surface area enclosed in same boundary

Low Temp:

$2^N$ possible ground states, for a Wilson loop with $P_S$ length

$$C_S = e^{KN_P}\left[1 + (3N - P_S)e^{-8\hat{K}}(+1) + (-1)P_S e^{-8\hat{K}} + \cdots\right]$$

unaffected; changes loop value

selecting a non-loop to flip; selecting loop to flip

At high temp decay is dominated by area of loop while at low temp it is dominated by length of loop

High temperature series can be used to reproduce gaussian model

First, consider a standard high Temp series expansion

$$Z = \sum_{\{\sigma_i\}} e^{K\sum_{\langle ij\rangle}\sigma_i\sigma_j} = 2^N \cosh^{dN} K \times S$$

↖ sum over allowed graphs weighted by $t^{\#\text{bond}}$

We can consider a related quantity $S'$ that refers to disconnected graphs

The key distinction between the two is that $S'$ allows for bond reuse while $S$ does not

In the $S'$ computation we can considered realizations of closed random walks

$$\ln S' = \sum \text{closed random walks} \times t^{\text{length of walk}}$$

We can set this up as a transfer matrix problem

$$\frac{\ln S'}{N} = \frac{1}{2}\sum_\ell \frac{t^\ell}{\ell} \langle 0|W(\ell)|0\rangle$$

↑ bidirectionality of random walks

↑ $1/\ell$ accounts for $\ell$ different starting points

$$\langle\sigma(0)\sigma(r)\rangle = \frac{1}{Z}\sum_{\{\sigma_i\}} \sigma(0)\sigma(r)\prod_{\langle ij\rangle}(1+t\sigma_i\sigma_j)$$

$$= \frac{\text{0}\!\sim\!\text{r} + \text{0}\!\sim\!\text{r} + \text{0}\!\sim\!\text{r}}{1 + \text{loop} + \text{loop}} = \text{0}\!\sim\!\text{r}$$

So the correlation length is effectively the phantom path

$$\approx \sum_\ell t^\ell \langle r|W(\ell)|0\rangle$$

We can now solve this as a random walk on a lattice with a Transition matrix

$$\langle r|W(\ell)|0\rangle = \sum_{r'} \langle r|W(1)|r'\rangle \times \langle r'|W(\ell-1)|0\rangle$$

$$W(\ell) = T^\ell$$

$$\langle r|T|r'\rangle = \begin{cases} 1 & r \text{ and } r' \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases}$$

We can then diagonalize the transition matrix by applying fourier transform w/ translational symmetry

$$\langle\sigma(r)\sigma(0)\rangle \approx \sum_\ell^\infty t^\ell \langle r|W(\ell)|0\rangle = \sum_\ell^\infty \langle r|(tT)^\ell|0\rangle$$

$$= \langle r\left|\frac{1}{1-tT}\right|0\rangle = \sum_q \langle r|q\rangle \frac{1}{1-tT(q)}\langle q|0\rangle$$

↑ geometric series

$$= \int \frac{d^d q}{(2\pi)^d} \frac{e^{iqr}}{1-2t\sum_\alpha \cos q_\alpha}$$

The first singularity occurs at $q=0$ $t_c$ defined $\frac{1}{T(0)} = 1/2d$

Below this temp small loops dominate and away from it large loops dominate

If we expand the denominator of the correlation function

$$1 - tT(q) = 1 - 2t \sum_{\alpha=1}^{d} \cos q_\alpha \simeq (1-2dt) + tq^2$$

$$\approx t_c \left( \xi^{-2} + q^2 + O(q^4) \right)$$

$$\xi = \left( \frac{1-2dt}{t_c} \right)^{-1/2}$$

This result recovers those found in the gaussian model

Free energy can be computed through a similar argument

$$\frac{\ln \delta'}{N} = \frac{1}{2} \sum_{\ell}^{\infty} \frac{t^\ell}{\ell} \langle 0 | W(\ell) | 0 \rangle = \frac{1}{2} \left\langle 0 \left| \sum_{\ell}^{\infty} \frac{(Tt)^\ell}{\ell} \right| 0 \right\rangle$$

$$= \langle 0 | \ln(1-tT) | 0 \rangle = -\frac{N}{2} \int \frac{d^d q}{(2\pi)^d} \langle 0|q\rangle \ln(1-tT(q)) \langle q|0\rangle$$

$$= -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \ln \left( 1 - 2t \sum_{\alpha=1}^{d} \cos q_\alpha \right)$$

The connection to gaussian critical dimension calculations is due to the no-intersection assumption of the phantom loop model

This assumption holds true for $d \geq 4$ but becomes increasingly probable in lower dimensions

## Exact Solution to Square Lattice Ising Model

The phantom loop computation w/ $\delta'$ is an overestimate of $\delta$

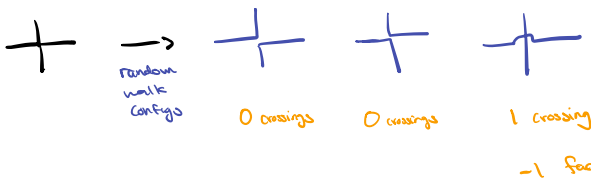
i) Overlapping sites can be modeled as multiple distinct random walks

ii) Phantom loop solution allows for random walkers to pass through bonds multiple times

In 2-D Ising Model we propose

$$\delta = \sum \text{loops of random walk w/o U-turns} \times t^{\#\text{ bonds}} \times (-1)^{\#\text{ crossings}}$$

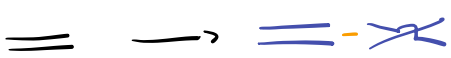
a) Overlap overcounting




The correction from the third config accounts for overcounting

This argument generalizes to all intersections

b) Multiplicity



Need to explicitly exclude U-turns

We can thus write

$$\delta = \sum (RW^* \text{ w/ 1 loop}) + \sum (RW^* \text{ w/ 2 loop}) + \cdots$$

* referring to no U turns

$$= \exp\left[\sum (RW^* \text{ w/ 1 loop})\right]$$

← only interactions from sign crossings so we can multiplicatively expand many loops (?)

Free Energy is consequently

$$\ln Z = N \ln 2 + 2N \cosh K + \sum (RW^* \text{ w/ 1 loop } \times t^{\#\text{bonds}})$$

$$\frac{\ln Z}{N} = \ln(2\cosh^2 K) + \sum_{\ell}^{\infty} \frac{t^\ell}{\ell} \langle 0|W^*(\ell)|0\rangle$$

$\langle 0|W^*(\ell)|0\rangle$ = # closed loops w/ $\ell$ steps, no U-turns from 0 to 0 $\times (-1)^{\#\text{ crossings}}$

The crossing # is no longer Markovian so we need special consideration

We use Whitney's theorem to compute $n_c$

$$(n_c) \bmod 2 = \left(1 + \frac{\Theta}{2\pi}\right) \bmod 2$$

Sum of tangent vectors along a loop should give us crossing #

We can compute this locally

$$(-1)^{n_c} = e^{i\pi n_c} = \exp\left[i\pi\left(1 + \frac{\Theta}{2\pi}\right)\right] = -e^{i/2 \sum_{j=1}^{\ell} \theta_j}$$

↑ angle contribution from $j^{th}$ step

$$T^* = \begin{bmatrix} \rightarrow\rightarrow & \rightarrow\uparrow & \rightleftharpoons & \rightarrow\downarrow \\ \uparrow\rightarrow & \uparrow\uparrow & \leftarrow\uparrow & \uparrow\downarrow \\ \rightleftharpoons & \uparrow\leftarrow & \leftarrow\leftarrow & \downarrow\leftarrow \\ \downarrow\rightarrow & \downarrow\uparrow & \leftarrow\downarrow & \downarrow\downarrow \end{bmatrix}$$

$$\langle x'y'|T^*|x,y\rangle = \begin{bmatrix} \langle x',y'|x+1,y\rangle & \langle x',y'|x+1,y\rangle e^{i\pi/4} & 0 & \langle x',y'|x+1,y\rangle e^{i\pi/4} \\ \langle x',y'|x,y+1\rangle e^{-i\pi/4} & \langle x',y'|x,y+1\rangle & \langle x',y'|x,y+1\rangle e^{i\pi/4} & 0 \\ 0 & \langle x',y'|x-1,y\rangle e^{-i\pi/4} & \langle x',y'|x-1,y\rangle & \langle x',y'|x-1,y\rangle e^{i\pi/4} \\ \langle x',y'|x,y-1\rangle e^{i\pi/4} & 0 & \langle x',y'|x,y-1\rangle e^{-i\pi/4} & \langle x',y'|x,y-1\rangle \end{bmatrix}$$

Matrix now keeps track of phase

This can be block diagonalized in Fourier basis

$$\sum_{xy} \langle x'y',u'|T^*|xy,u\rangle \langle xy|q_x q_y\rangle = \langle u'|T^*(q)|u\rangle \langle x'y'|q_x q_y\rangle$$

$$T_q^* = \begin{bmatrix} e^{-iq_x} & e^{-i(q_x-\pi/4)} & 0 & e^{-i(q_x+\pi/4)} \\ e^{-i(q_y+\pi/4)} & e^{-iq_y} & e^{-i(q_y-\pi/4)} & 0 \\ 0 & e^{i(q_x-\pi/4)} & e^{iq_x} & e^{i(q_x+\pi/4)} \\ e^{i(q_y+\pi/4)} & 0 & e^{i(q_y-\pi/4)} & e^{iq_y} \end{bmatrix}$$

We can now sum over rw's and directions

$$\langle 0|w^*(Q)|0\rangle = \sum_{\mu=1}^{4} (00,\mu|T^{*\ell}|00,\mu) = \frac{1}{N}\sum_{xy,\mu}(xy,\mu|T^{*\ell}|xy,\mu) = \frac{1}{N}\mathrm{tr}(T^{*\ell})$$

Free Energy follows as

$$\frac{\ln Z}{N} = \ln(2\cosh^2 K) - \frac{1}{2}\sum_{\ell}\frac{t^\ell}{\ell}\langle 0|w^*(Q)|0\rangle = \ln(2\cosh^2 K) - \frac{1}{2N}\mathrm{tr}\left[\sum_{\ell}\frac{T^{*\ell}}{\ell}\right]$$

$$= \ln(2\cosh^2 K) + \frac{1}{2N}\ln(1-tT^*)$$

$$\frac{1}{2N}\sum_{q}\mathrm{tr}\ln(1-tT(q))$$

$$= \ln(2\cosh^2 K) + \frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\ln\left[(1+t)^2 - 2t(1-t^2)(\cos q_x + \cos q_y)\right]$$

$$= \ln 2 + \frac{1}{2}\int_{-\pi}^{\pi}\frac{dq_x\,dq_y}{(2\pi)^2}\ln\left[\cosh^2(2K) - \sinh(2K)(\cos q_x + \cos q_y)\right]$$

## 8. Beyond Spin Waves

In discrete spin models low-temperature expansions rely on flipping spins out of the bulk; however in continuous spin models low-temp excitations take the form of goldstone modes

We know that these excitations destroy long-range order when $d \leq 2$ but for $d$ near 2 goldstone modes can provide a mechanism for understanding criticality via low-temp expansions

### Nonlinear $\sigma$ model

Consider $n$ component spins on a lattice

$$S(i) = (S_1, S_2, \ldots S_n) \qquad |S(i)|^2 = S_1^2 + S_2^2 + \cdots + S_n^2 = 1$$

The nearest neighbor Hamiltonian can be written

$$-\beta\mathcal{H} = K\sum_{\langle ij\rangle} S(i)\cdot S(j) = K\sum_{\langle ij\rangle}\left(1 - \frac{(S(i)-S(j))^2}{2}\right)$$

At low temp the neighboring fluctuations can be approximated by the gradient

$$-\beta\mathcal{H} = -\beta E_0 - \frac{K}{2}\int d^dx\,(\nabla S(x))^2$$

Ignore ground state energy

$$Z = \int \mathcal{D}[S(x)]\,\delta(S(x)-1)\,e^{-K/2\int d^dx\,(\nabla S(x))^2}$$

Any constant spin vector produces a ground state configuration

Consider $S = (0, 0, \ldots, 1)$

There are $n-1$ goldstone modes for transverse fluctuations

More generally, ($n$-component vector)

$$S(x) = (\pi_1(x), \pi_2(x), \ldots \pi_{n-1}(x), \sigma(x)) = (\vec{\pi}(x), \sigma(x))$$

We can cleverly rewrite the $|S| = 1$ constraint as follows

$$\int d\vec{s}\, \delta(s^2 - 1) = \int_{-\infty}^{\infty} d\pi\, d\sigma\, \delta(\pi^2 + \sigma^2 - 1)$$

$$= \int_{-\infty}^{\infty} d\pi\, d\sigma\, \delta\left[(\sigma - \sqrt{1-\pi^2})(\sigma + \sqrt{1-\pi^2})\right]$$

$\delta(ax) = \frac{\delta(x)}{|a|}$

$$= \int_{-\infty}^{\infty} d\pi\, d\sigma\, \frac{\delta\left[(\sigma - \sqrt{1-\pi^2}\right]}{\sigma + \sqrt{1-\pi^2}}$$

$$= \int_{-\infty}^{\infty} \frac{d\pi}{2\sqrt{1-\pi^2}}$$

The partition function can thus be written

$$Z \propto \int \frac{D\pi(x)}{2\sqrt{1-\pi^2}}\, e^{-\frac{K}{2}\int d^d x \left[(\nabla\pi)^2 + (\nabla\sqrt{1-\pi^2})^2\right]}$$

$$= \int D\pi(x)\, \exp\left\{ -\int d^d x\, \frac{K}{2}(\nabla\pi)^2 + \frac{K}{2}\left(\nabla\sqrt{1-\pi^2}\right)^2 + \frac{\rho}{2}\ln(1-\pi^2)\right\}$$

$$\rho = N/V = 1/a^d$$

The nonlinear terms in this Hamiltonian reflect symmetry constraints of $\vec{\pi}$

We can accordingly expand around $\vec{\pi}(x)$

$$\mathcal{H}[\vec{\pi}(x)] = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots$$

$$\mathcal{H}_0 = \frac{K}{2}\int d^d x\, (\nabla\pi)^2 \quad \leftarrow \text{Independent Goldstone Modes}$$

$$\mathcal{H}_1 = \int d^d x \left[\frac{K}{2}(\pi\cdot\nabla\pi)^2 - \frac{\rho}{2}\pi^2\right]$$

$\nabla\sqrt{1-\pi^2}$ $\quad -\ln(1-\pi^2) = \pi^2 + \frac{\pi^4}{2} + \ldots$

In fourier space the expansion is

$$\mathcal{H}_0 = \frac{K}{2}\int \frac{d^d q}{(2\pi)^d}\, q^2 |\pi(q)|^2$$

$$\mathcal{H}_1 = -\frac{K}{2}\int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\, \pi_\alpha(q_1)\pi_\alpha(q_2)\pi_\beta(q_3)\pi_\beta(-q_1-q_2-q_3)(q_1\cdot q_2)$$

$$-\frac{\rho}{2}\int \frac{d^d q}{(2\pi)^d}\, |\pi(q)|^2$$

From quadratic theory we can write the correlation functions of Goldstone Modes

$$\langle \pi_\alpha(q)\pi_\beta(q')\rangle_0 = \frac{\delta_{\alpha\beta}(2\pi)^d \delta^d(q+q')}{Kq^2}$$ (Non-interacting goldstone modes)

The corresponding real space fluctuations are

$$\langle \pi(x)^2\rangle_0 = \int \frac{d^dq}{(2\pi)^d}\langle|\pi(q)|^2\rangle_0 = \frac{(n-1)}{K}\int_{1/L}^{1/a}\frac{d^dq}{(2\pi)^d}\cdot\frac{1}{q^2}$$

$$= \frac{(n-1)}{K}\,\frac{K_d\left(a^{2-d} - L^{2-d}\right)}{(d-2)}$$

When $d \le 2$ the fluctuations diverge as $L \to \infty$

This inspired Polyakov to compute RG in expansions of T (Not fully seeing the logic here)

$$\mathcal{BH}[\pi^<] = V\mathcal{F}_b^0 + \mathcal{BH}_0[\pi^<] + \langle U_1[\pi^< + \pi^>]\rangle_0 + O(T^2)$$

Integrating over short momentum $\pi^>$

(Skipping Diagrams and perturbation computation)

$$= V\mathcal{F}_b^0 + V\mathcal{F}_b^1 + \frac{K}{2}\left(1 + \frac{I_d(b)}{K}\right)\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,q^2\,|\vec{\pi}^<(q)|^2$$

$$+ \frac{K}{2}\int \frac{d^dq_1\,d^dq_2\,d^dq_3}{(2\pi)^{3d}}\,\pi_\alpha^<(q_1)\pi_\alpha^<(q_2)\,\pi_\beta^<(q_3)\,\pi_\beta^<(-q_1-q_2-q_3)\,(q_1\cdot q_2)$$

$$- \frac{p}{2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,|\pi^<(q)|^2 \times \left[1-(1-b^{-d})\right]$$

To compute the rescaling factor $\zeta$ we consider the length of the spin after short mode averaging

$$\langle \vec{s}\rangle^> = \left\langle \left(\pi_1^< + \pi_1^>, \ldots, \sqrt{1-(\pi^<+\pi^>)^2}\right)\right\rangle_0^>$$

$$= \left(\pi_1^<, \ldots, 1 - \frac{(\pi^<)^2}{2} - \left\langle\frac{(\pi^>)^2}{2}\right\rangle_0^> + \ldots\right)$$

$$= \left(1 - \left\langle\frac{(\pi^>)^2}{2}\right\rangle_0^> + O(T^2)\right)\left(\pi_1^<, \ldots, \sqrt{1-(\pi^<)^2}\right)$$

Our rescaling is thus

$$\zeta = 1 - \left\langle\frac{(\pi^>)^2}{2}\right\rangle_0^2 + O(T^2) = 1 - \frac{(n-1)}{2}\,\frac{I_d(b)}{K} + O(T^2)$$

Skipped detailed calculations for rest of RG