

Classical Mechanics Derivations

Poisson's Equation

Solve for gravitational flux Φ_m

$$\Phi_m = \int_S \mathbf{n} \cdot \mathbf{g} \, da = - \int_S \frac{Gm \cos \theta}{r^2} \, da = -4\pi G \int_V \rho \, dV$$

using
steradians

$$\mathbf{n} \cdot \mathbf{g} = -Gm \frac{\cos \theta}{r^2}$$

Gauss's Divergence Theorem:

$$\int_S \mathbf{n} \cdot \mathbf{g} \, da = \int_V \nabla \cdot \mathbf{g} \, dV$$

$$\int_V -4\pi G \rho \, dV = \int_V \nabla \cdot \mathbf{g} \, dV \quad \text{for arbitrary } V \text{ so } -4\pi G \rho = \nabla \cdot \mathbf{g}$$

substituting $\mathbf{g} = -\nabla \Phi$

$$\boxed{\nabla^2 \Phi = 4\pi G \rho}$$

Euler's Equation

Goal: Find extremum of $J = \int_{x_1}^{x_2} f(y(x), y'(x); x) \, dx = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x); x) \, dx$

$$y(\alpha, x) = y(0, x) + \alpha \eta(x)$$

Consider $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad \text{and} \quad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \frac{d\eta}{dx} \right) dx$$

Integration by parts:

$$\int u \, dv = uv - \int v \, du \quad u = \frac{\partial f}{\partial y'} \quad v = \eta(x) \, dx$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx} \, dx = \left. \frac{\partial f}{\partial y'} \cdot \eta(x) \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \, dx \cdot \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$$

$$\overbrace{\quad} = 0$$

Since $\eta(x_2) = \eta(x_1)$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right) \eta(x) dx$$

if $\frac{\partial J}{\partial \alpha} = 0$ $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$ must vanish since $\eta(x)$ is arbitrary

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0}$$

Euler's Equation w/ Constraints

Consider $f = f\{y, y', z, z'; x\}$ and $g\{y, z; x\} = 0$

$$dg = \left(\frac{\partial g}{\partial y} \cdot \frac{dy}{d\alpha} + \frac{\partial g}{\partial z} \cdot \frac{dz}{d\alpha} \right) d\alpha = 0$$

$\frac{dx}{d\alpha}$ vanishes

$$y(\alpha, x) = y(x) + \alpha \eta_1(x)$$

$$z(\alpha, x) = z(x) + \alpha \eta_2(x)$$

$$\frac{dy}{d\alpha} = \eta_1(x) \quad \text{and} \quad \frac{dz}{d\alpha} = \eta_2(x)$$

Plugging in to $\frac{dg}{d\alpha} = 0$

$$\frac{\partial g}{\partial y} \eta_1(x) = - \frac{\partial g}{\partial z} \eta_2(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \eta_2(x) \right] dx$$

Factor out η_1 with $\frac{\eta_2(x)}{\eta_1(x)} = \frac{\frac{\partial g}{\partial y}}{-\frac{\partial g}{\partial z}}$

$$= \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \left(\frac{\partial g / \partial y}{\partial g / \partial z} \right) \right] \eta_1(x) dx$$

$\eta_1(x)$ is arbitrary so

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} \right) \left(\frac{\partial g}{\partial z} \right)^{-1}$$

Each side of the equation is equal to a function of x : $-\lambda(x)$

Thus,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \lambda(x) \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} + \lambda(x) \frac{\partial g}{\partial z} = 0$$

↑
Lagrange
undetermined
multiplier

Generalized:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y'_i} + \sum_j \lambda_j(x) \frac{\partial g_j}{\partial y_i} = 0$$

Conservation of Energy

A closed system cannot depend explicitly on time

$$\frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j$$

Lagrange's Equation:

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}$$

Plugging Lagrange's Equation into $\frac{dL}{dt}$

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \ddot{q}_j} \ddot{q}_j$$

Using chain rule:

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0$$

$$\frac{d}{dt} \left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = 0$$

Let

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H$$

If potential energy is independent of $\dot{x}_{\alpha,i}$ or t then, $U = U(q_j)$ and $\frac{\partial U}{\partial \dot{q}_j} = 0$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T - U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

Therefore,

$$(T - U) - \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = -H$$

From an earlier (excluded) proof $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$

$$T - U - 2T = -H$$

$$\boxed{T + U = E = H = \text{constant}}$$

Conservation of Linear Momentum

For an infinitesimal translation of every radius vector $r_\alpha \rightarrow r_\alpha + \delta r$

Where $L = L(x_i, \dot{x}_i)$, $\delta r = \sum_i \delta x_i e_i$ changes L to

$$\delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i = 0$$

For an infinitesimal translation

$$\delta \dot{x}_i = \delta \frac{dx_i}{dt} = \frac{d}{dt} \delta x_i = 0$$

$$\delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x_i = 0$$

Since δx_i is independent of displacement, δL only vanishes when $\frac{\partial L}{\partial x_i} = 0$

Lagrange's equation then becomes

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0$$

Therefore,

$$\frac{\partial L}{\partial \dot{x}_i} \text{ is constant}$$

$$\frac{\partial (T - U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \left(\frac{1}{2} m \sum_i \dot{x}_i^2 \right) = m \dot{x}_i = p_i = \text{constant}$$

Conservation of Angular Momentum

Mechanical properties of a closed system are unaffected by orientation of the system

For an infinitesimal angle shift $\delta\theta$

$$\delta \mathbf{r} = \delta\theta \times \mathbf{r}$$

$$\delta \dot{\mathbf{r}} = \delta\theta \times \dot{\mathbf{r}}$$

$$\delta L = \sum_i \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i + \sum_i \frac{\partial L}{\partial x_i} \delta x_i = 0$$

As proven above:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \text{so} \quad \dot{p}_i = \frac{\partial L}{\partial x_i}$$

$$\delta L = \sum_i \dot{p}_i \delta x_i + \sum_i p_i \delta \dot{x}_i = 0$$

$$\dot{\mathbf{p}} \cdot \delta \mathbf{r} + \mathbf{p} \cdot \delta \dot{\mathbf{r}} = 0$$

$$\dot{\mathbf{p}} (\delta\theta \times \mathbf{r}) + \mathbf{p} (\delta\theta \times \dot{\mathbf{r}}) = 0$$

$$\delta\theta \cdot (\mathbf{r} \times \mathbf{p}) + \delta\theta \cdot (\dot{\mathbf{r}} \times \mathbf{p}) = 0$$

$$\delta\theta \cdot [\mathbf{r} \times \mathbf{p} + (\dot{\mathbf{r}} \times \mathbf{p})] = 0$$

Rewriting the expression

$$\delta\theta \cdot \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = 0$$

Since $\delta\theta$ is arbitrary we can simplify this expression

$$\boxed{\frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = 0}$$

Kepler's Second Law

Consider a particle of mass m moving in a central force field

We know that angular momentum is conserved due to spherical symmetry

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{constant} \quad \text{so} \quad \mathbf{r} \text{ and } \mathbf{p} \text{ are normal to } \mathbf{L}$$

Lagrangian:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

Since Lagrangian is cyclic in θ

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}$$

Therefore,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{constant}$$

Let $L = m r^2 \dot{\theta}$

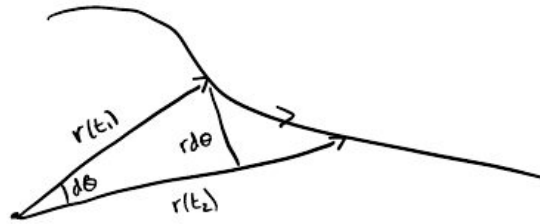
Area traced out:

$$dA = \frac{1}{2} r^2 d\theta$$

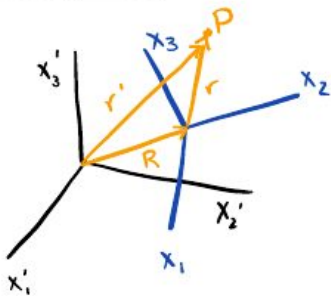
Areal Velocity:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$= \frac{L}{2m} = \text{constant}$$



Rotating Axis



$x' \rightarrow$ fixed axis

$x \rightarrow$ rotating axis

$$r' = R + r$$

For infinitesimal rotation of x_i system

$$(dr)_{\text{fixed}} = d\theta \times r$$

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \frac{d\theta}{dt} \times r$$

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \omega \times r$$

If P is rotating as well

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \left(\frac{dr}{dt}\right)_{\text{rotating}} + \omega \times r$$

For r' :

$$\left(\frac{dr'}{dt}\right)_{\text{fixed}} = \left(\frac{dR}{dt}\right)_{\text{fixed}} + \left(\frac{dr}{dt}\right)_{\text{fixed}}$$

$$= \left(\frac{dR}{dt}\right)_{\text{fixed}} + \left(\frac{dr}{dt}\right)_{\text{rotating}} + \omega \times r$$

Let $V_f = \left(\frac{dr}{dt} \right)_{\text{fixed}}$: Velocity relative to fixed axis

$V = \left(\frac{dR}{dt} \right)_{\text{fixed}}$: Linear velocity of rotating axis relative to fixed axis

$V_r = \left(\frac{dr}{dt} \right)_{\text{rotating}}$: Velocity relative to rotating axis

ω : Angular velocity of rotating axis

$$V_f = V + V_r + \omega \times r$$

Centrifugal and Coriolis Forces

Newton's 2nd Law only applies in inertial reference frames

$$F = ma = m \left(\frac{dV_f}{dt} \right)_{\text{fixed}}$$

$$= m \left(\underbrace{\left(\frac{dV}{dt} \right)_{\text{fixed}}}_{\ddot{R}_f} + \underbrace{\left(\frac{dV_r}{dt} \right)_{\text{fixed}}}_{\left(\frac{dV_r}{dt} \right)_f = \left(\frac{dV_r}{dt} \right)_R + \omega \times V_r = a_r + \omega \times V_r} + \underbrace{\omega \times r + \omega \times \left(\frac{dr}{dt} \right)_{\text{fixed}}}_{= \omega \times \left(\frac{dr}{dt} \right)_{\text{rotating}} + \omega \times (\omega \times r) = \omega \times V_r + \omega \times (\omega \times r)} \right)$$

$$F = m\ddot{R}_f + ma_r + m\omega \times r + 2m\omega \times V_r + m\omega \times (\omega \times r)$$

However, in the noninertial reference frame, the effective force

$$F_{\text{eff}} = ma_r$$

$$= F - m\ddot{R}_f - m\omega \times r - m\omega \times (\omega \times r) - 2m\omega \times V_r$$

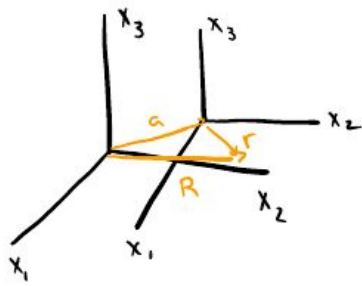
↑
translational
acceleration

↑
rotational
acceleration

↑
Centrifugal Force
($= m\omega^2 r$ when
 $\omega \perp r$)

↑
Coriolis Force

Parallel Axis Theorem



$$R = a + r$$

$$X_i = a_i + x_i$$

$$J_{ij} = \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} X_{\alpha, \kappa}^2 - X_{\alpha, i} X_{\alpha, j} \right)$$

$$= \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} (X_{\alpha, \kappa} + a_{\kappa})^2 - (X_{\alpha, i} + a_i)(X_{\alpha, j} + a_j) \right)$$

$$= \underbrace{\sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} X_{\alpha, \kappa}^2 - X_{\alpha, i} X_{\alpha, j} \right)}_{I_{ij}} + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} (2X_{\alpha, \kappa} a_{\kappa} + a_{\kappa}^2) - (a_i X_{\alpha, j} + a_j X_{\alpha, i} + a_i a_j) \right)$$

$$= I_{ij} + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} a_{\kappa}^2 - a_i a_j \right) + \sum_{\alpha} m_{\alpha} \left(2 \delta_{ij} \sum_{\kappa} X_{\alpha, \kappa} a_{\kappa} - a_i X_{\alpha, j} - a_j X_{\alpha, i} \right)$$

$$\sum_{\alpha} m_{\alpha} r_{\alpha} = 0 \quad \text{since the center of mass is located at } 0$$

$$\sum_{\alpha} m_{\alpha} r_{\alpha, \kappa} = 0$$

$$J_{ij} = I_{ij} + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{\kappa} a_{\kappa}^2 - a_i a_j \right)$$

$$J_{ij} = I_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

Lorentz Transformations

Assume the most simple coordinate transformation to account for Galilean shortcomings

$$x'_1 = \gamma(x_1 - vt) \quad \text{and} \quad x_1 = \gamma'(x'_1 + vt')$$

Laws of physics must be equivalent in each reference frame so $\gamma' = \gamma$

$$x'_1 = \gamma(x_1 - vt) \quad \text{and} \quad x_1 = \gamma(x'_1 + vt')$$

$$x_1 = \gamma(\gamma(x_1 - vt) + vt') = \gamma^2(x_1 - vt) + \gamma vt'$$

$$t' = \frac{x_1 - \gamma^2(x_1 - vt)}{\gamma v} = \frac{x_1}{\gamma v} - \frac{\gamma x_1}{v} + \gamma t = \frac{x_1}{\gamma v} (1 - \gamma^2) + \gamma t$$

Speed of light must be constant in each reference frame

$$x_1 = ct$$

$$x'_1 = ct'$$

$$x_1 = \gamma(x'_1 + vt') = \gamma(x'_1 + \frac{v}{c}x'_1)$$

$$\frac{x_1}{x'_1} = \gamma \left(1 + \frac{v}{c}\right)$$

$$\Rightarrow \frac{x'_1}{x_1} = \frac{1}{\gamma(1 + \frac{v}{c})}$$

$$\frac{1}{\gamma(1 + \frac{v}{c})} = \gamma \left(1 - \frac{v}{c}\right)$$

$$1 = \gamma^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x'_1 = \gamma(x_1 - vt) = \gamma \left(x_1 - \frac{v}{c}x_1\right)$$

$$\frac{x'_1}{x_1} = \gamma \left(1 - \frac{v}{c}\right)$$

Full Transformation:

$$x'_1 = \gamma(x_1 - vt) = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

$$t' = \frac{t - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Length Contraction



Length measured:

$$L' = X'_1(2) - X'_1(1) = \frac{X_1(2) - v t(2)}{\sqrt{1 - v^2/c^2}} - \frac{X_1(1) - v t(1)}{\sqrt{1 - v^2/c^2}} = \frac{(X_1(2) - X_1(1)) - v(t(2) - t(1))}{\sqrt{1 - v^2/c^2}}$$

$$t(2) = t(1) \quad \frac{t(1) - \frac{v X_1(1)}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{t(2) - \frac{v X_1(2)}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$t(2) - t(1) = \frac{v}{c^2} (X_1(2) - X_1(1))$$

$$L' = \frac{L - \frac{v^2}{c^2} L}{\sqrt{1 - v^2/c^2}} = L \left(\frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - v^2/c^2}} \right) = L \sqrt{1 - v^2/c^2}$$

Time Dilation (fixed position)

$$\Delta t = t(2) - t(1)$$

Via Lorentz transformations

$$\Delta t' = t'(2) - t'(1) = \frac{t(2) - \frac{v X_1(2)}{c^2}}{\sqrt{1 - v^2/c^2}} - \frac{t(1) - \frac{v X_1(1)}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{t(2) - t(1)}{\sqrt{1 - v^2/c^2}} \quad X_1(2) = X_1(1)$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

Relativistic Energy

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = T_2 - T_1 \quad \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt} (\gamma m \mathbf{u})$$

$$W = T = \int \frac{d}{dt} (\gamma m u) \cdot u dt = m \int_0^u u \, d(\gamma u)$$

Integration by parts

$$T = \gamma m u^2 - m \int_0^u \frac{u du}{\sqrt{1 - u^2/c^2}} = \gamma m c^2 - m c^2$$

Let $E_0 = mc^2$ represent rest energy

$$E = \gamma mc^2 = T + E_0$$

↑
Sum of kinetic
and other energy

Relativistic Momentum and Energy

$$p = \gamma mu$$

$$p^2 c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2} \right)$$

$$\left(\frac{u^2}{c^2} \right) = 1 - \frac{1}{\gamma^2}$$

$$\begin{aligned} p^2 c^2 &= \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2} \right) = \gamma^2 m^2 c^4 - m^2 c^4 \\ &= E^2 - E_0^2 \end{aligned}$$

$E^2 = p^2 c^2 + E_0^2$

Spacetime Interval

Define Δs^2 s.t. it is invariant in all inertial systems in relative motion

$$\Delta s^2 = \sum_{j=1}^3 (\Delta x_j)^2 - c^2 \Delta t^2 \quad \Rightarrow \quad ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2$$

$$\Delta s^2 = \Delta s'^2 = \sum_{j=1}^3 (\Delta x'_j)^2 - c^2 \Delta t'^2$$